CHAPTER 9 – RATIONAL EQUATIONS AND FUNCTIONS

9.1 – INVERSE AND JOINT VARIATION

DIRECT VARIATION

- Direct variation occurs if $y = kx$, where k is a constant.
- To check for direct variation, divide the output by the input. Direct variation will always yield the same ratio.

$217 \div 31 = 7$
$140 \div 20 = 7$
$119 \div 17 = 7$
$84 \div 12 = 7$

$y = 7x$

input **x**	output **y**
31	217
20	140
17	119
12	84

INVERSE VARIATION

- Inverse variation occurs if $\boxed{y = \frac{k}{x}}$, where k is a non-zero constant.
- To check for ~~direct~~ inverse variation, multiply the output by the input. Inverse variation will always yield the same product.

$1.5 \times 20 = 30$
$2.5 \times 12 = 30$
$4 \times 7.5 = 30$
$5 \times 6 = 30$

$$y = \frac{30}{x}$$

x (input)	y (output)
1.5	20
2.5	12
4	7.5
5	6

CLASSIFYING DIRECT AND INVERSE VARIATION

GIVEN EQUATION	REWRITTEN EQUATION	TYPE OF VARIATION
a. $\frac{y}{5} = x$	$y = 5x$	direct, $k=5$
b. $y = x + 2$	cannot be rewritten	neither
c. $xy = 4$	$y = \frac{4}{x}$	inverse variation

CLASSIFYING DIRECT AND INVERSE VARIATION

GIVEN EQUATION	REWRITTEN EQUATION	TYPE OF VARIATION
$xy = \frac{1}{4}$	$y = \frac{1}{4x}$	inverse
$\frac{x}{y} = 5$ $x = 5y$	$y = \frac{1}{5}x$ $y = \frac{x}{5}$	direct
$y = x - 3$		neither
$\frac{1}{2}xy = 9$	$y = \frac{18}{x}$	inverse

WRITING INVERSE VARIATION EQUATIONS

The variables x and y vary inversely, and $y = 8$ when $x = 3$.

a. Write an equation that relates x and y. → find k

b. Find y when $x = -4$.

$$8 = \frac{k}{3}$$

$$24 = k$$

$$y = \frac{24}{-4}$$

$$y = -6$$

WRITING INVERSE VARIATION EQUATIONS

INVERSE VARIATION MODELS The variables x and y vary inversely. Use the given values to write an equation relating x and y. Then find y when $x = 2$.

29. $x = 5, y = -2$

$-2 = \frac{k}{5}$

$-10 = k$

$y = \frac{-10}{x}$

$y = \frac{-10}{2} = -5$

30. $x = 4, y = 8$

$8 = \frac{k}{4}$

$32 = k$

$y = \frac{32}{x}$

$y = \frac{32}{x} = 16$

31. $x = 7, y = 1$

$1 = \frac{k}{7}$

$7 = k$

$y = \frac{7}{x}$

$y = \frac{7}{2} = 3.5$

JOINT VARIATION

- Joint variation occurs when a quantity varies directly as the product of two or more other quantities. $z = kxy$

" z varies jointly with x and y"

JOINT VARIATION

turn into "x =" form

Tell whether x varies jointly with y and z.

12. $x = 15yz$

yes

13. $\frac{x}{z} = 0.5y$

$x = 0.5yz$

yes

14. $xy = 4z$

$x = \frac{4z}{y}$

No

15. $x = \frac{yz}{2}$

$= \frac{1}{2}yz$

yes

JOINT VARIATION MODELS The variable z varies jointly with x and y. Use the given values to write an equation relating x, y, and z. Then find z when $x = -4$ and $y = 7$.

$z = kxy$

39. $x = 3, y = 8, z = 6$

$6 = k(3)(8) = 24k$

$\frac{6}{24} = k \rightarrow k = \frac{1}{4}$

$z = \frac{xy}{4}$ $\quad z = \frac{(4)(7)}{4} = 7$

40. $x = -12, y = 4, z = 2$

$2 = k(-12)(4)$

$-\frac{2}{48} = k$

$-\frac{1}{24} = k$

$z = \frac{-xy}{24}$

$z = \frac{-(-4)(7)}{24} = \frac{7}{6}$

41. $x = 1, y = \frac{1}{3}, z = 5$

$5 = k(1)(\frac{1}{3})$

$15 = k$

$z = 15xy$

$z = 15(-4)(7) = -420$

42. $x = -6, y = 3, z = \frac{2}{5}$

$\frac{2}{5} = k(-6)(3)$

$-\frac{1}{45} = k$

$z = \frac{-xy}{45} = \frac{(-4)(7)}{45}$

$z = \frac{28}{45}$

Write an equation for the given relationship.

directly: var. at top
inversely: var at bottom

RELATIONSHIP	EQUATION
a. y varies directly with x.	$y = kx$
b. y varies inversely with x.	$y = \frac{k}{x}$
c. z varies jointly with x and y.	$z = kxy$
d. y varies inversely with the square of x.	$y = \frac{k}{x^2}$
e. z varies directly with y and inversely with x.	$z = \frac{ky}{x}$

9.2 – GRAPHING SIMPLE RATIONAL EQUATIONS

BASIC RATIONAL FUNCTION

- A **rational function** is of the form $\boxed{f(x) = \frac{p(x)}{q(x)}}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

because we cannot divide by 2.

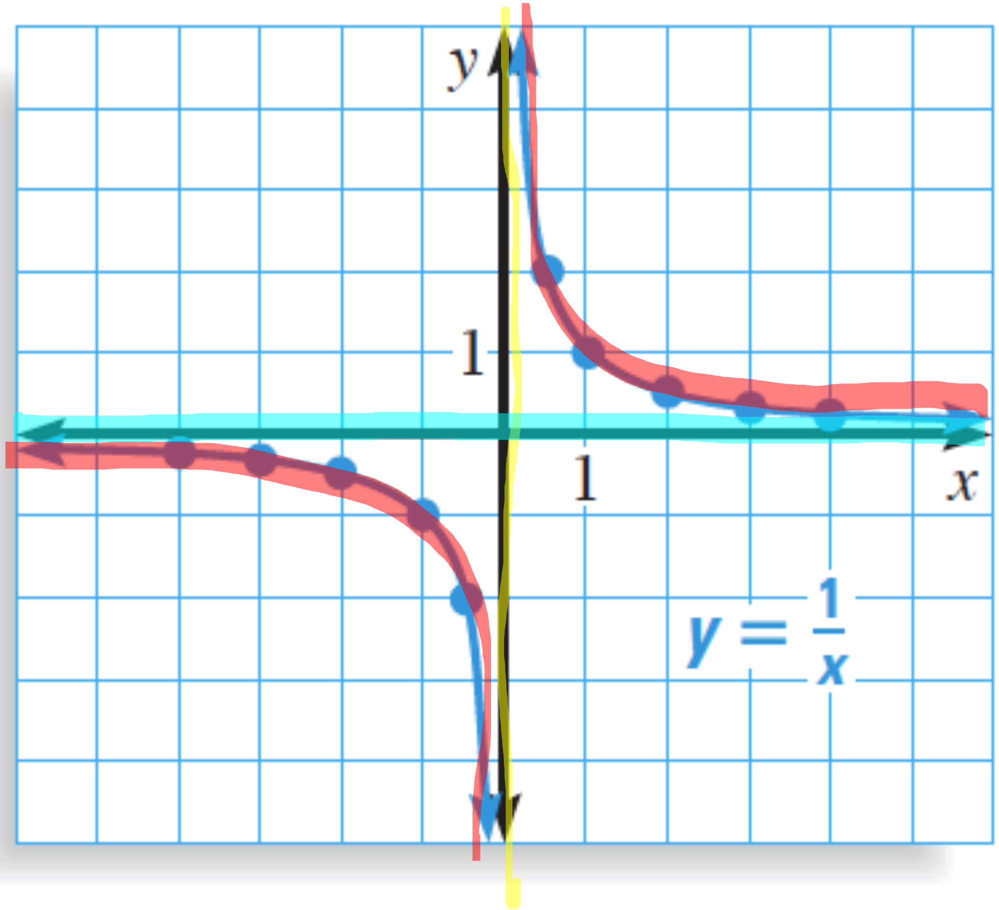
- It has two parts called **branches.**

BASIC RATIONAL FUNCTION

- A **rational function** where the top and bottom polynomial are both linear (first degree). $ax+b$
- The graph is called a **hyperbola**.
- It has a horizontal and a vertical asymptote.
- It has two parts called **branches.**
- Domain and range are all real number except for the values of the asymptotes.

$f(x) = \frac{1}{x}$ Domain and range are all real numbers except for x = 0 and y = 0.

x	y
-4	$-\frac{1}{4}$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2



x	y
4	$\frac{1}{4}$
3	$\frac{1}{3}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2

RATIONAL FUNCTION WITH TRANSFORMATIONS

- Is of the form $f(x) = \frac{a}{x-h} + k$ or $f(x) = \frac{ax+b}{cx+d}$.
- In the form $f(x) = \frac{a}{x-h} + k$, the asymptotes are $x = h$ and $y = k$.
- In the form $f(x) = \frac{ax+b}{cx+d}$, the asymptotes are $x = \frac{-d}{c}$ and $y = \frac{a}{c}$.

$cx+d \neq 0$

$cx+d=0$

$\frac{cx}{c} = \frac{-d}{c}$ $\quad x = \frac{-d}{c}$

GRAPHING RATIONAL FUNCTIONS WITH TRANSFORMATIONS

- 1) Find and graph the asymptotes.
- 2) Graph two points on each side of the vertical asymptote.

Graph $y = \frac{-2}{x+3} - 1$. State the domain and range.

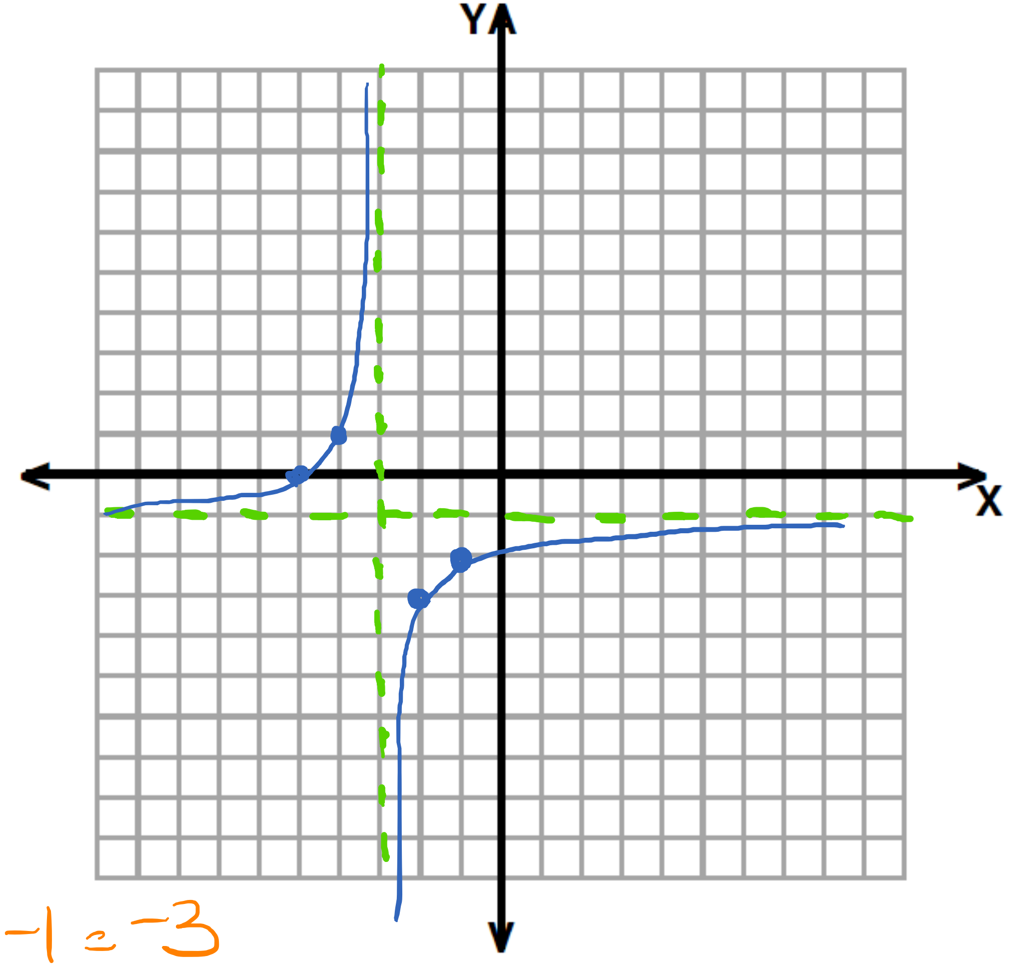
$x + 3 = 0$

$x = -3$

1) Asymptotes

$h = -3 \rightarrow x = -3$

$k = -1 \rightarrow y = -1$



2) Find points

x	y
-1	-2
-2	-3
-3	
-4	1
-5	0

$y = \frac{-2}{(-1)+3} - 1 = \frac{-2}{2} - 1$

$= -2$

$y = \frac{-2}{(-2)+3} - 1 = \frac{-2}{1} - 1 = -3$

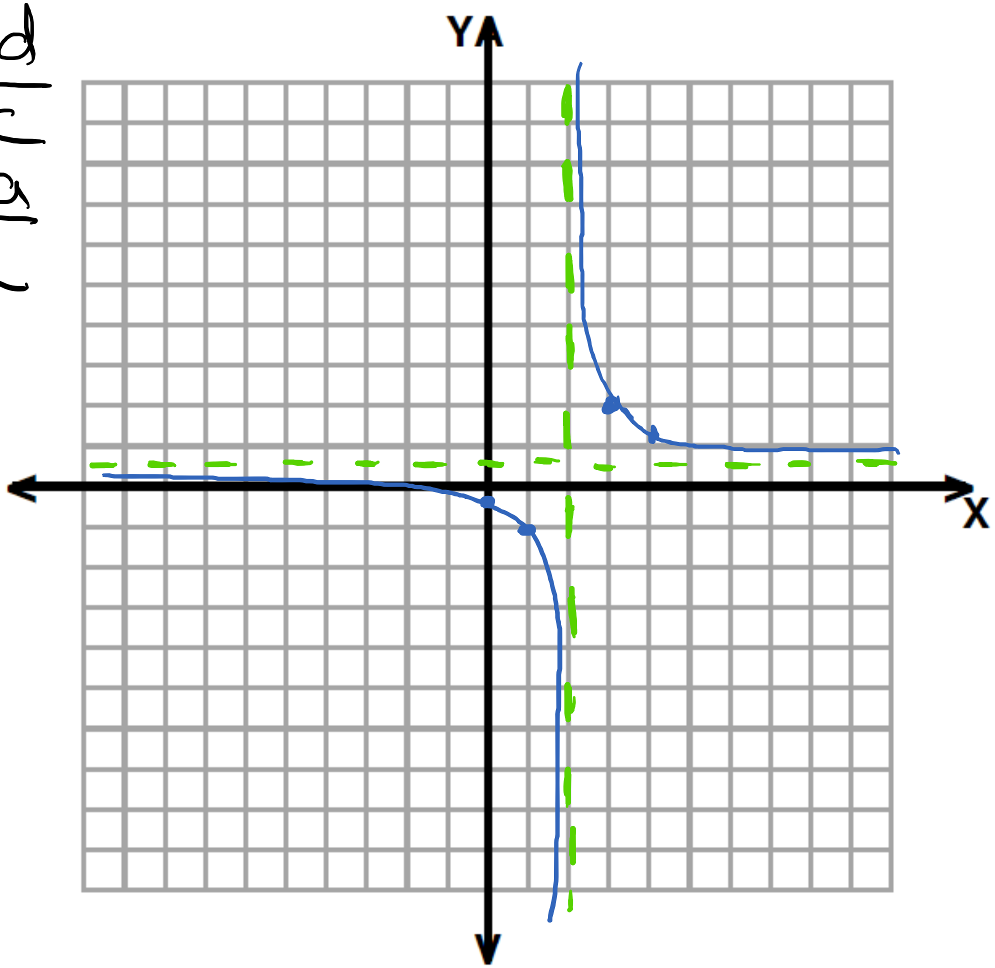
$y = \frac{-2}{(-4)+3} - 1 = \frac{-2}{-1} - 1 = 1$

$y = \frac{-2}{(-5)+3} - 1 = \frac{-2}{-2} - 1 = 0$

Graph $y = \dfrac{1x + 1}{2x - 4}$. State the domain and range.

$x = \dfrac{-d}{c}$

$Y = \dfrac{a}{c}$



1) Asymptotes

$x = \dfrac{4}{2} = 2 \qquad y = \dfrac{1}{2}$

2)

x	y
0	$-\frac{1}{4}$
1	-1
2	
3	2
4	1.25

$y = \dfrac{0+1}{2(0)-4} = -\dfrac{1}{4}$

$y = \dfrac{1+1}{2(1)-4} = \dfrac{2}{-2} = -1$

$y = \dfrac{3+1}{2(3)-4} = \dfrac{4}{2} = 2$

$y = \dfrac{4+1}{2(4)-4} = \dfrac{5}{4} = 1.25$

9.3 – GRAPHING GENERAL RATIONAL FUNCTIONS

CHARACTERISTICS OF GENERAL RATIONAL FUNCTION

For a **rational function** of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$:

- The x-intercepts of the graph are the real zeros of $p(x)$.
  $y=0$
- The graph has vertical asymptotes at all the real zeros of $q(x)$.

HORIZONTAL ASYMPTOTES OF GENERAL RATIONAL FUNCTION

For a **rational function** of the form

coefficient   constant

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$$

HORIZONTAL ASYMPTOTES OF GENERAL RATIONAL FUNCTION

Polynomial characteristic	Asymptote / End behavior
Bottom polynomial has higher degree. $y=\frac{x}{x^3}$	$y = 0$ asymptote
$y=\frac{2}{3}$ $y=\frac{2x^2+3x}{3x^2-4}$ Polynomials have same degree.	$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$ asymptote $y = \frac{a_m}{b_n}$
Top polynomial has higher degree.	$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$ end behavior. $y = \frac{a_m}{b_n} x^{m-n}$

$y=\frac{2x^3}{3x}$ $y=\frac{2x^2}{3}$ → touchdown.

GRAPHING RATIONAL FUNCTIONS ~~WITH TRANSFORMATIONS~~

horizontal (look at chart)

- 1) Find and graph the asymptotes. → vertical (bottom = top)

make bottom equal to zero and solve.

- 2) Find and graph the zeros. → top = 0
- 3) Find and graph some points around the asymptotes.

→ make table of values.

Graph $y = \frac{4}{x^2 + 1}$. State the domain and range.

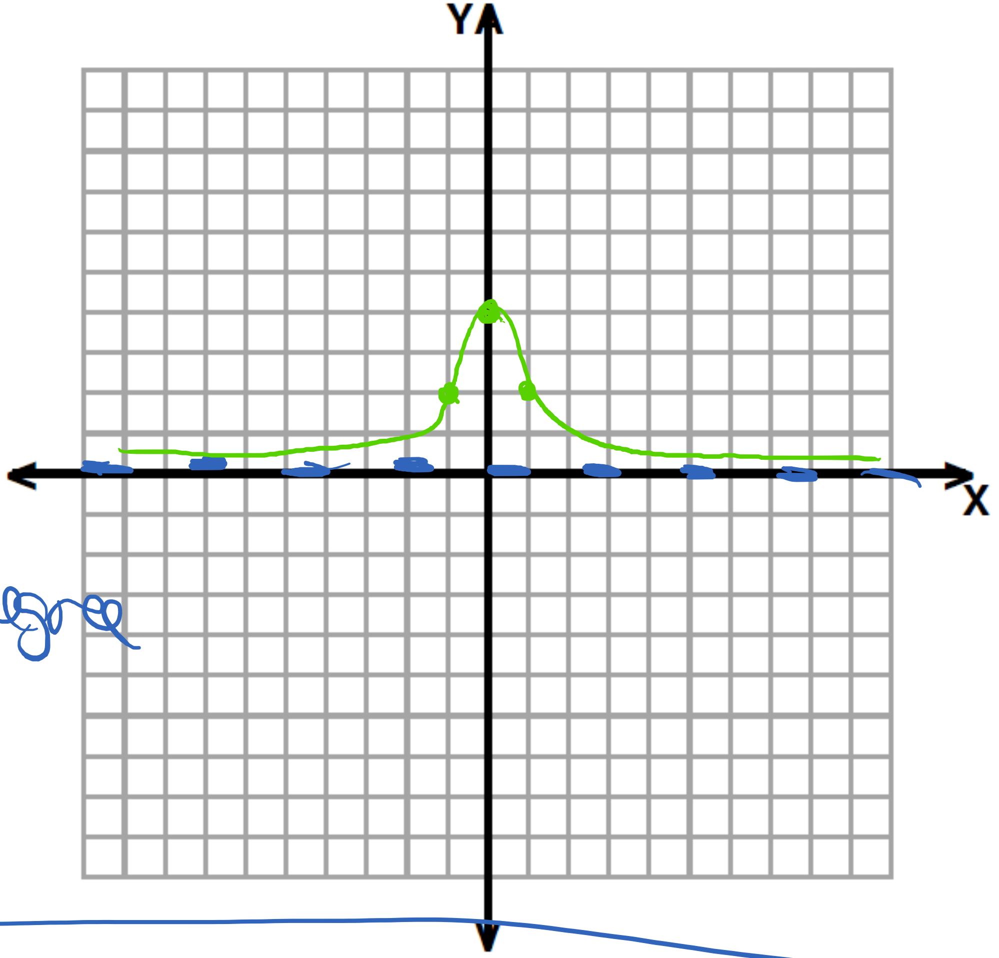
1) vertical $x^2+1=0$
will never happen
→ no vertical asymptote

horizontal: bottom has higher degree
→ $y=0$ is asymptote.

2) $4 \neq 0$
→ no zeros

3)

x	y
-1	2
0	4
1	2



$x=0 \quad \frac{4}{0^2+1} = 4$

$x=-1 \quad \frac{4}{(-1)^2+1} = \frac{4}{2} = 2$

$x=1 \quad \frac{4}{1^2+1} = \frac{4}{2} = 2$

Graph $y = \dfrac{3x^2}{x^2 - 4}$.

1) asymptotes

vertical $x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2 \quad x = 2$

horizontal

same degree

$y = \frac{3}{1} \quad y = 3$

2) zeros: $3x^2 = 0$

$x = 0$

3)

x	y
-4	4
-3	5.4
$x = -2$	
-1	-1
0	0
1	-1
$x = 2$	
3	5.4
4	4

$x = -4$

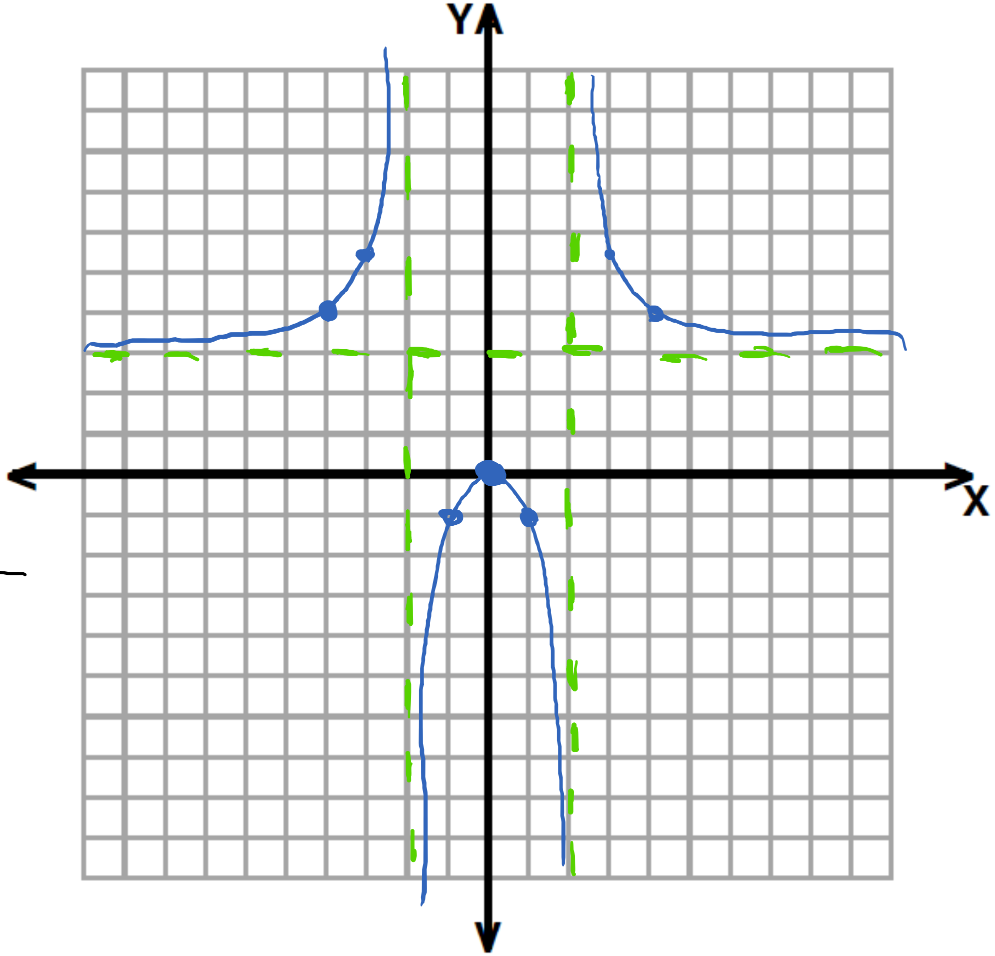
$\dfrac{3(-4)^2}{(-4)^2 - 4} = \dfrac{48}{12} = 4$

$x = -3$

$\dfrac{3(-3)^2}{(-3)^2 - 4} = \dfrac{27}{5} = 5.4$

$x = -1$

$\dfrac{3(-1)^2}{(-1)^2 - 4} = \dfrac{3}{-3} = -1$



Graph $y = \dfrac{x^2 - 2x - 3}{x + 4}$.

1) Vertical $x + 4 = 0$

$x = -4$

horizontal end behavior $y = \dfrac{x^2}{x} = x$

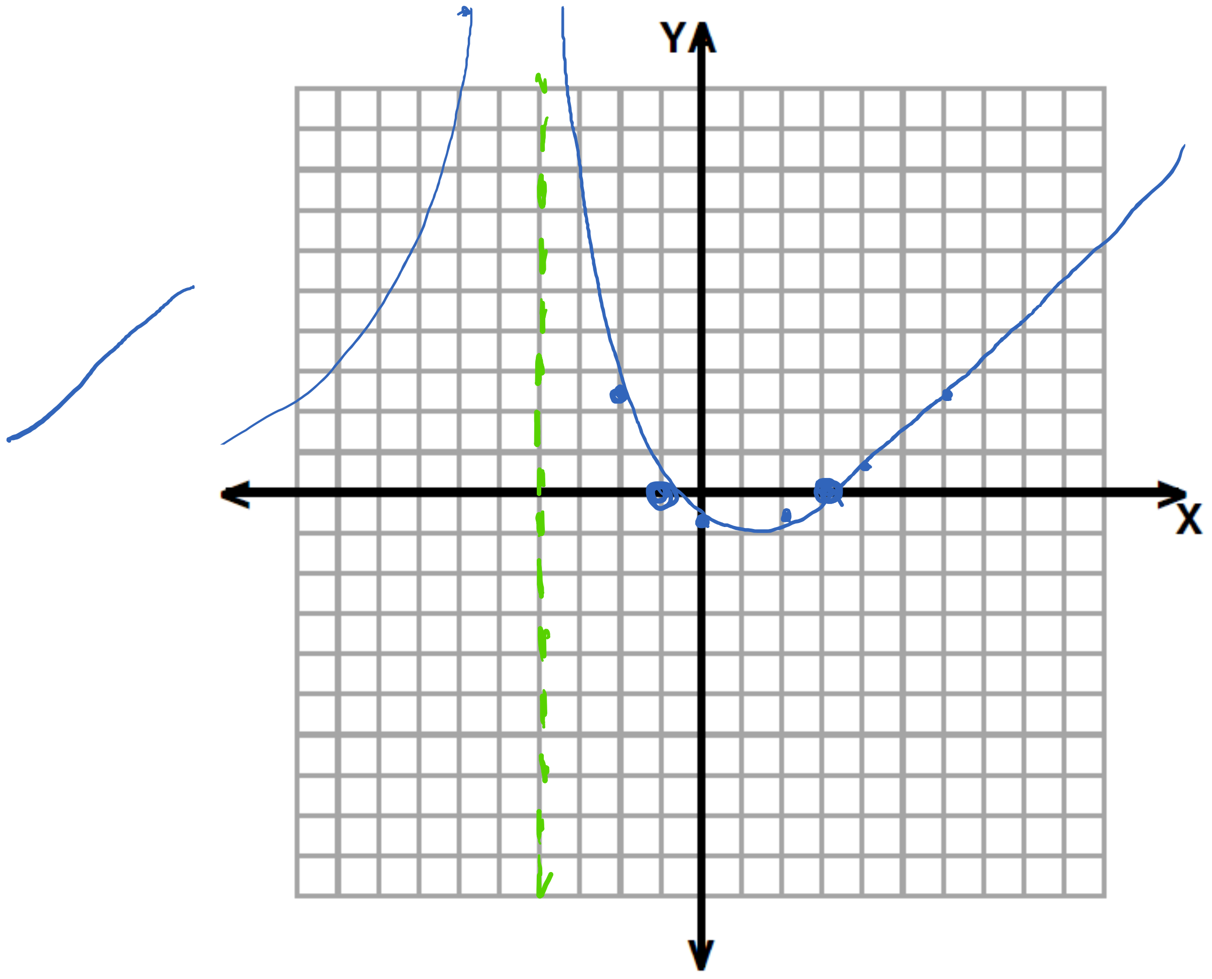
2) Zeros $x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

3)

x	y
-9	-19.2
-6	-22.5
$x = -4$	
-2	2.5
0	-0.75
2	-0.5
4	0.63
6	2.1



9.4 – MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS

SIMPLIFYING A RATIONAL EXPRESSION

Simplify: $\dfrac{x^2 - 4x - 12}{x^2 - 4}$ $= \dfrac{(x-6)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$

$\dfrac{6}{4} = \dfrac{\cancel{2}\cdot 3}{\cancel{2}\cdot 2} = \dfrac{3}{2}$

$\dfrac{x-6}{x+2}$

MULTIPLYING RATIONAL EXPRESSIONS

Multiply: $\dfrac{4x - 4x^2}{x^2 + 2x - 3} \cdot \dfrac{x^2 + x - 6}{4x}$

$$\frac{\cancel{4x}(1-x)}{\cancel{(x+3)}(x-1)} \cdot \frac{\cancel{(x+3)}(x-2)}{\cancel{4x}}$$

$$\frac{\cancel{-(x-1)}}{\cancel{x-1}} \cdot (x-2)$$

$$-(x-2) = -x+2$$

$$\frac{7}{3} \cdot \frac{2}{5} = \frac{14}{15}$$

$$\frac{\overset{7}{\cancel{14}}}{3} \cdot \frac{2}{\underset{5}{\cancel{10}}} = \frac{14}{15}$$

MULTIPLYING RATIONAL EXPRESSIONS BY A POLYNOMIAL

Multiply: $\dfrac{x+3}{8x^3-1} \cdot \dfrac{(4x^2+2x+1)}{1}$

$$\frac{3}{2} \cdot \frac{4}{1} = \frac{12}{2}$$

$$6$$

$$\frac{x+3}{(2x-1)\cancel{(4x^2+2x+1)}} \cdot \frac{\cancel{4x^2+2x+1}}{1}$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$\frac{x+3}{2x-1}$$

DIVIDING RATIONAL EXPRESSIONS

Divide: $\dfrac{5x}{3x - 12} \div \dfrac{x^2 - 2x}{x^2 - 6x + 8}$

$$\frac{5x}{3(x-4)} \times \frac{x^2-6x+8}{x^2-2x}$$

$$\frac{5\cancel{x}}{3\cancel{(x-4)}} \times \frac{\cancel{(x-4)}\cancel{(x-2)}}{\cancel{x}\cancel{(x-2)}} = \frac{5}{3}$$

$$\frac{3}{4} \div \frac{7}{5}$$

$$\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$$

MULTIPLYING AND DIVIDING

Divide: $\dfrac{6x^2 + 7x - 3}{6x^2} \div (2x^2 + 3x)$

$$\frac{(3x-1)\cancel{(2x+3)}}{6x^2} \times \frac{1}{x\cancel{(2x+3)}}$$

$$\frac{3x-1}{6x^3}$$

	$2x$	3
$3x$	$6x^2$	$9x$
-1	$-2x$	-3

$6x^2 \cdot (-3)$

$\underline{9x} \times \underline{-2x} = -18x^2$

$\underline{9x} + \underline{-2x} = 7x$

9.5 – ADDITION, SUBTRACTION AND COMPLEX FRACTIONS

ADDING / SUBTRACTING RATIONAL EXPRESSIONS WITH SAME DENOMINATORS

Perform the indicated operation.

$$\frac{2x}{x+3} - \frac{4}{x+3} = \frac{2x-4}{x+3}$$

$$= \frac{2(x-2)}{x+3}$$

$$\frac{3}{5} + \frac{2}{5} = \frac{5}{5} = 1$$

ADDING RATIONAL EXPRESSIONS WITH DIFFERENT DENOMINATORS

Add: $\dfrac{5}{6x^2} + \dfrac{x}{4x^2 - 12x} = \dfrac{5 \cdot 2(x-3)}{6x^2 \cdot 2(x-3)} + \dfrac{x \cdot 3x}{4x(x-3) \cdot 3x}$

$\dfrac{3 \cdot 3}{3 \cdot 4} + \dfrac{1 \cdot 2}{6 \cdot 2}$ LCM 12

$\dfrac{9}{12} + \dfrac{2}{12} = \dfrac{11}{12}$

LCM = $12x^2(x-3)$

$$\frac{5(2(x-3))}{6x^2 \cdot (2(x-3))} + \frac{x(3x)}{4x(x-3)(3x)}$$

$$\frac{10x - 30 + 3x^2}{12x^2(x-3)} = \frac{3x^2 + 10x - 30}{12x^2(x-3)}$$

SUBTRACTING RATIONAL EXPRESSIONS WITH DIFFERENT DENOMINATORS

Subtract: $\dfrac{x+1}{x^2+4x+4} - \dfrac{2}{x^2-4}$ $= \dfrac{x+1}{(x+2)^2} - \dfrac{2}{(x+2)(x-2)}$

LCM

$(x+2)^2(x-2)$

$$\frac{(x+1)(x-2) - 2(x+2)}{(x+2)^2(x-2)} = \frac{x^2 - x - 2 - 2x - 4}{(x+2)^2(x-2)}$$

$$= \frac{x^2 - 3x - 6}{(x+2)^2(x-2)}$$

SIMPLIFYING COMPLEX FRACTIONS

Simplify: $\dfrac{\frac{2}{x+2}}{\left(\frac{1}{x+2}+\frac{2}{x}\right)}$ $= \dfrac{2}{x+2} \div \left(\dfrac{1}{x+2} + \dfrac{2}{x}\right)$

$\dfrac{1}{x+2} + \dfrac{2}{x}$ LCM $x(x+2)$

$\dfrac{x \cdot 1 + 2(x+2)}{x(x+2)} = \dfrac{3x+2}{x(x+2)}$

$\dfrac{2}{x+2} \div \dfrac{3x+2}{x(x+2)} = \dfrac{2}{\cancel{x+2}} \times \dfrac{x\cancel{(x+2)}}{3x+2}$

$= \dfrac{2x}{3x+2}$

COMPLEX FRACTIONS "CHEAT SHEET"

- $\dfrac{\frac{a}{b}}{c} = \dfrac{a}{bc}$

$$\frac{\frac{a}{b}}{c} = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}$$

- $\dfrac{a}{\frac{b}{c}} = \dfrac{ac}{b}$

$$\frac{a}{\frac{b}{c}} = a \div \frac{b}{c} = \frac{a}{1} \times \frac{c}{b} = \frac{ac}{b}$$

- $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

9.6 – SOLVING RATIONAL EQUATIONS

STEPS TO SOLVING

- 1) Find the least common denominator of all the rational parts.
- 2) Multiply both sides by the LCD.
- 3) Solve the remaining equation.
- 4) Check for extraneous solutions.

EQUATION WITH ONE SOLUTION

Solve: $\frac{4}{x} + \frac{5}{2} = -\frac{11}{x}$

1) $2x$

$$2x\left(\frac{4}{x} + \frac{5}{2}\right) = -\frac{11}{x} \cdot 2x$$

$$2\cancel{x} \cdot \frac{4}{\cancel{x}} + \cancel{2}x \cdot \frac{5}{\cancel{2}} = -\frac{11}{\cancel{x}} \cdot 2\cancel{x}$$

$$8 + 5x = -22$$
$$-8 \qquad\qquad -8$$

$$5x = -30$$
$$x = -6$$

Plug back in:

$$\frac{4}{-6} + \frac{5}{2} = -\frac{11}{-6}$$

$\boxed{\text{Solution } x = -6}$

EQUATION WITH AN EXTRANEOUS SOLUTION

Solve: $\dfrac{5x}{x-2} = \dfrac{7}{1} + \dfrac{10}{x-2}$

LCD: $x-2$

$\cancel{(x-2)}\dfrac{5x}{\cancel{x-2}} = 7(x-2) + \dfrac{10}{\cancel{x-2}}\cancel{(x-2)}$

$5x = 7x - 14 + 10$

$5x = 7x - 4$

$-2x = -4$

$x = 2$

Plug back in

$\dfrac{5(2)}{2-2} = 7 + \dfrac{10}{2-2}$

0

2 is extraneous

No solution

EQUATION WITH TWO SOLUTIONS

Solve: $\dfrac{4x+1}{x+1} = \dfrac{12}{x^2-1} + 3$

$LCD = (x+1)(x-1)$

$$\frac{4x+1}{x+1} = \frac{12}{(x+1)(x-1)} + \frac{3}{1}$$

$$\frac{\cancel{(x+1)}(x-1)(4x+1)}{\cancel{x+1}} = \frac{12\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}} + 3(x+1)(x-1)$$

$$4x^2 - 3x - 1 = 12 + 3x^2 - 3$$

$$x^2 - 3x - 10 = 0$$

$$(x+2)(x-5) = 0$$

$$\boxed{x = -2 \quad x = 5}$$

$-2+1 \neq 0$ $\quad$ $5+1 \neq 0$

$(-2)^2 - 1 \neq 0$ $\quad$ $(5)^2 - 1 \neq 0$

SOLVING BY CROSS-MULTIPLYING

Solve: $\dfrac{2}{x^2 - x} = \dfrac{1}{x-1}$

↳ when you have one fraction bar on either side

$2(x-1) = 1(x^2 - x)$

$2x - 2 = x^2 - x$

$0 = x^2 - 3x + 2$

$0 = (x-2)(x-1)$

$x = 2$ ✓ $\quad \cancel{x = 1}$

↳ extraneous

$2^2 - 2 \neq 0$

$2 - 1 \neq 0$

$1^2 - 1 = 0$

$1 - 1 = 0$

solution: $x = 2$