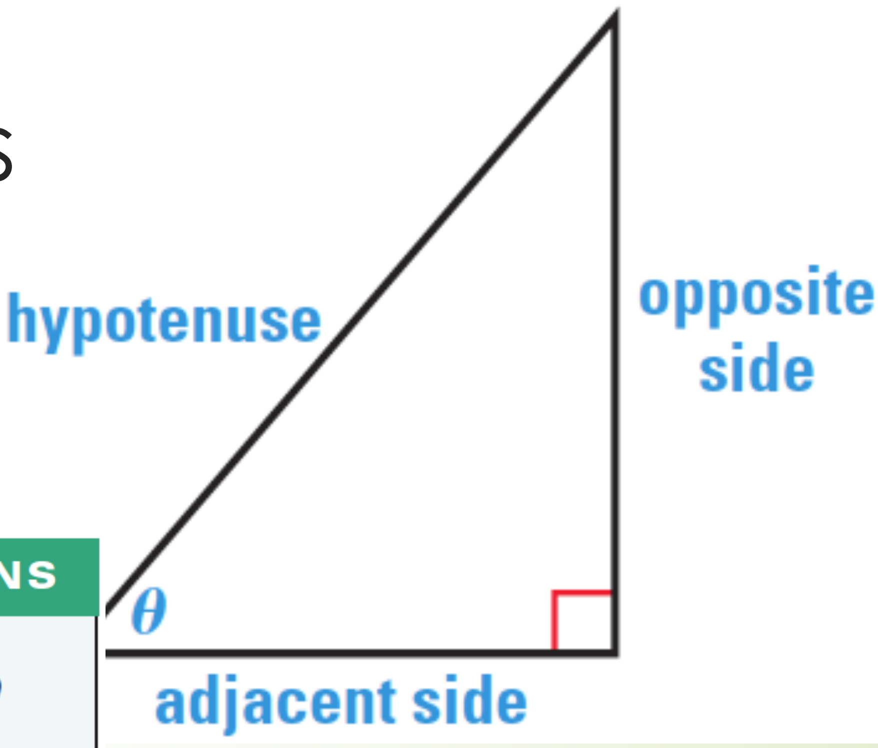
Chapter 13: Trigonometric Ratios and Functions

13.1 – Right Triangle Trigonometry

Recap of trigonometric ratios



RIGHT TRIANGLE DEFINITION OF TRIGONOMETRIC FUNCTIONS

Let θ be an acute angle of a right triangle. The six trigonometric functions of θ are defined as follows.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

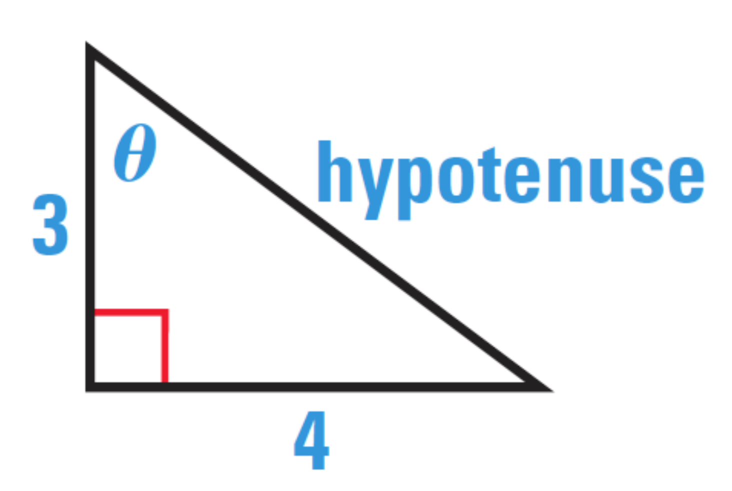
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations *opp, adj,* and *hyp* represent the lengths of the three sides of the right triangle. Note that the ratios in the second row are the reciprocals of the ratios in the first row. That is:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

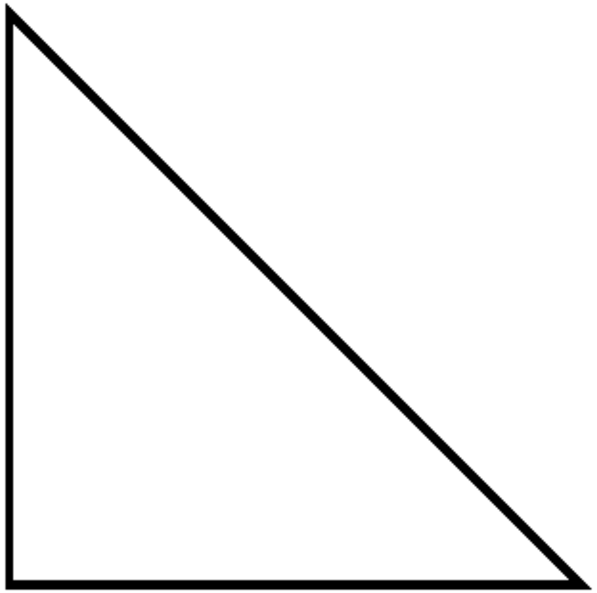
Evaluating Trigonometric Ratios

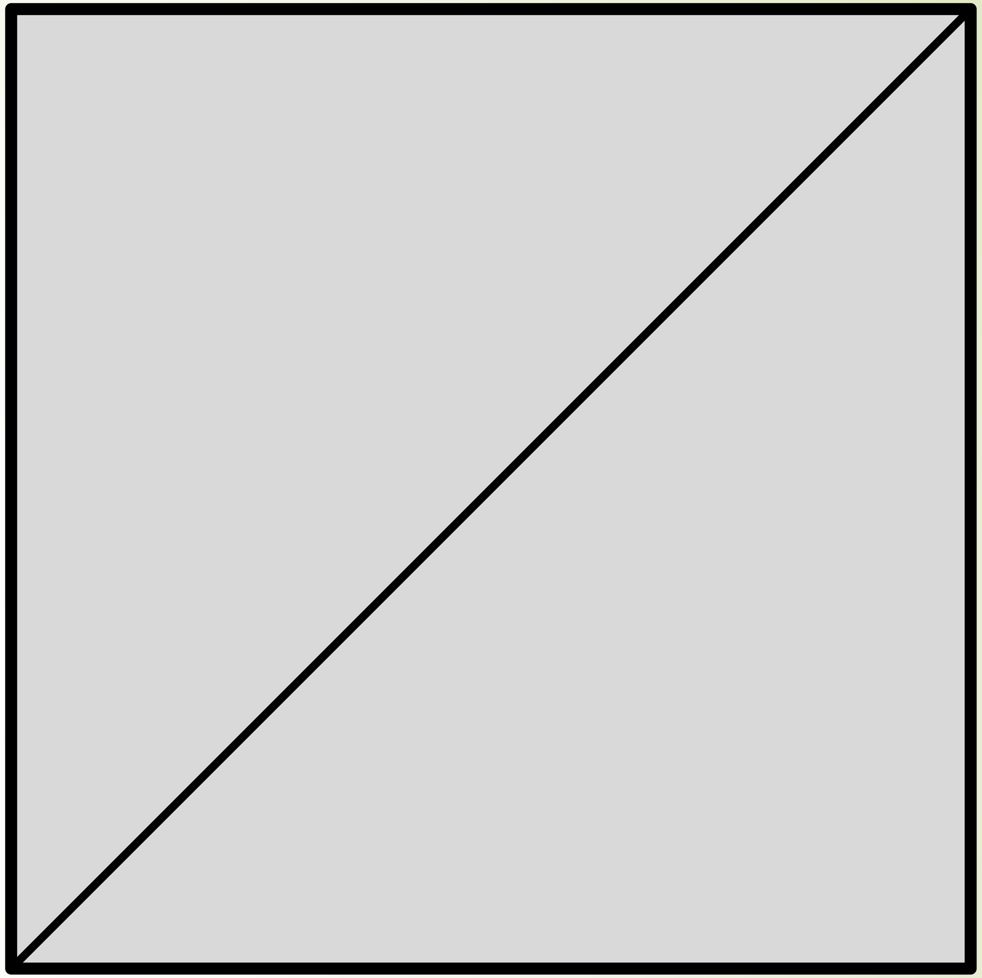
Evaluate the six trigonometric functions of the angle θ shown in the right triangle.



Special Right Triangle

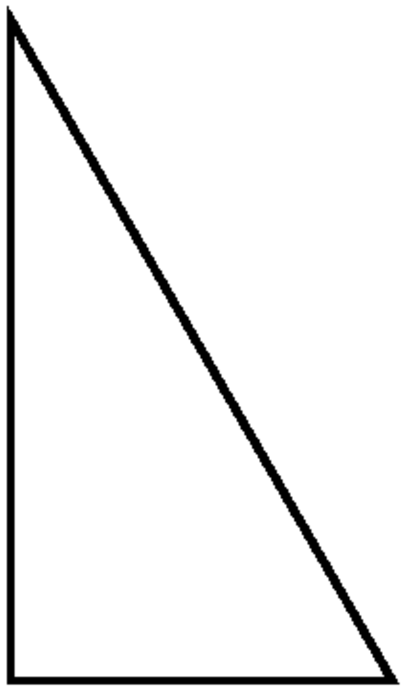
Angle Measures		
Side Measures		

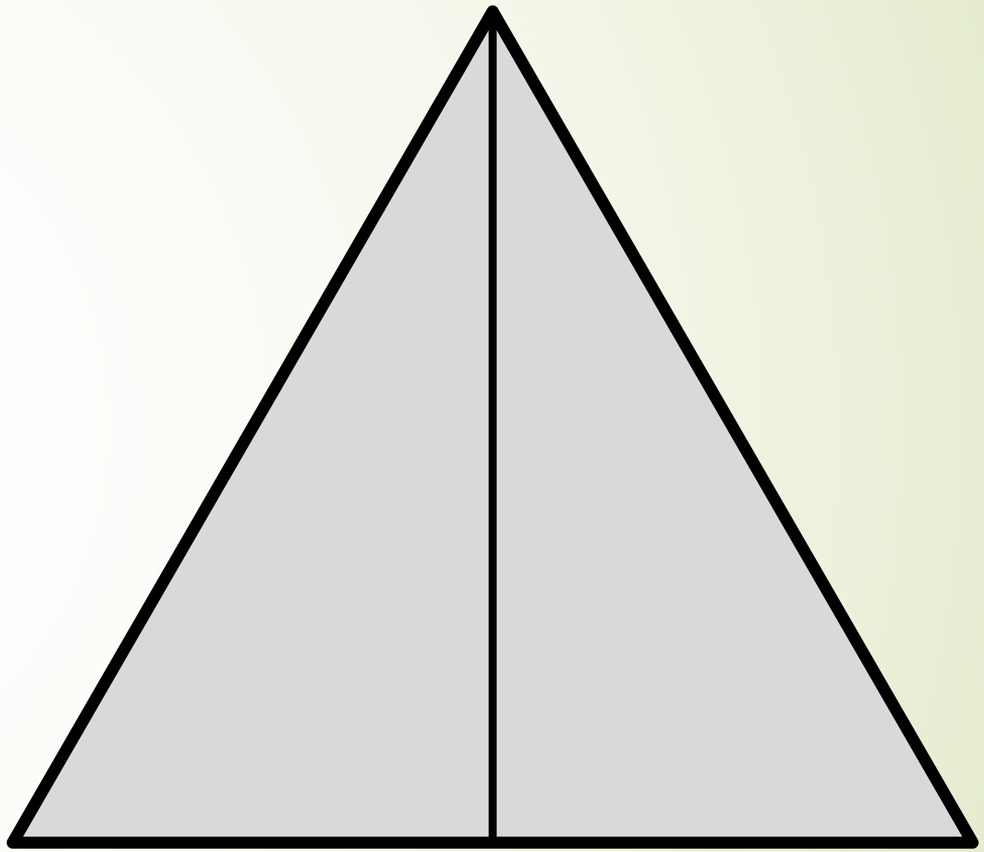




Special Right Triangle

Angle Measures		
Side Measures		



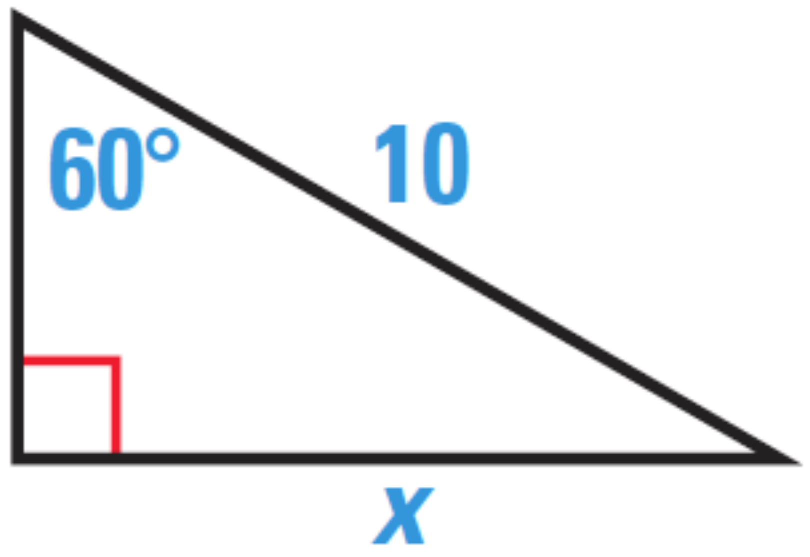


Ratios of Common Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Find the value of x for the right triangle shown.

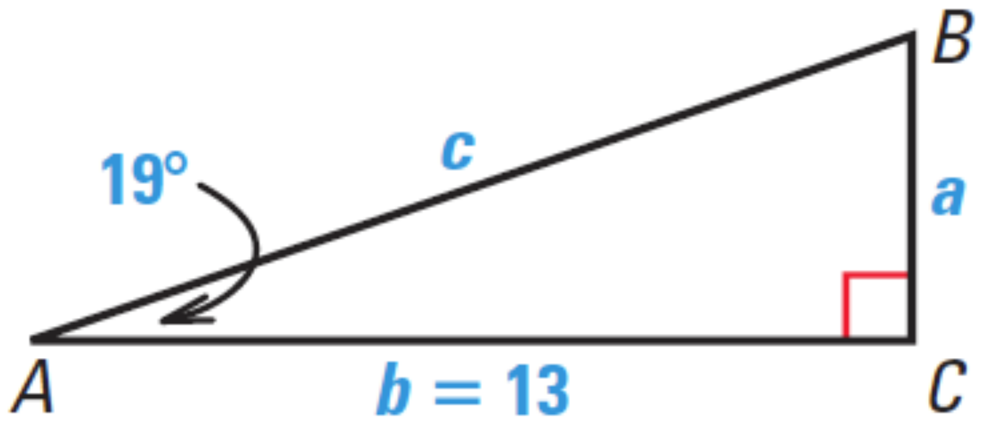
8



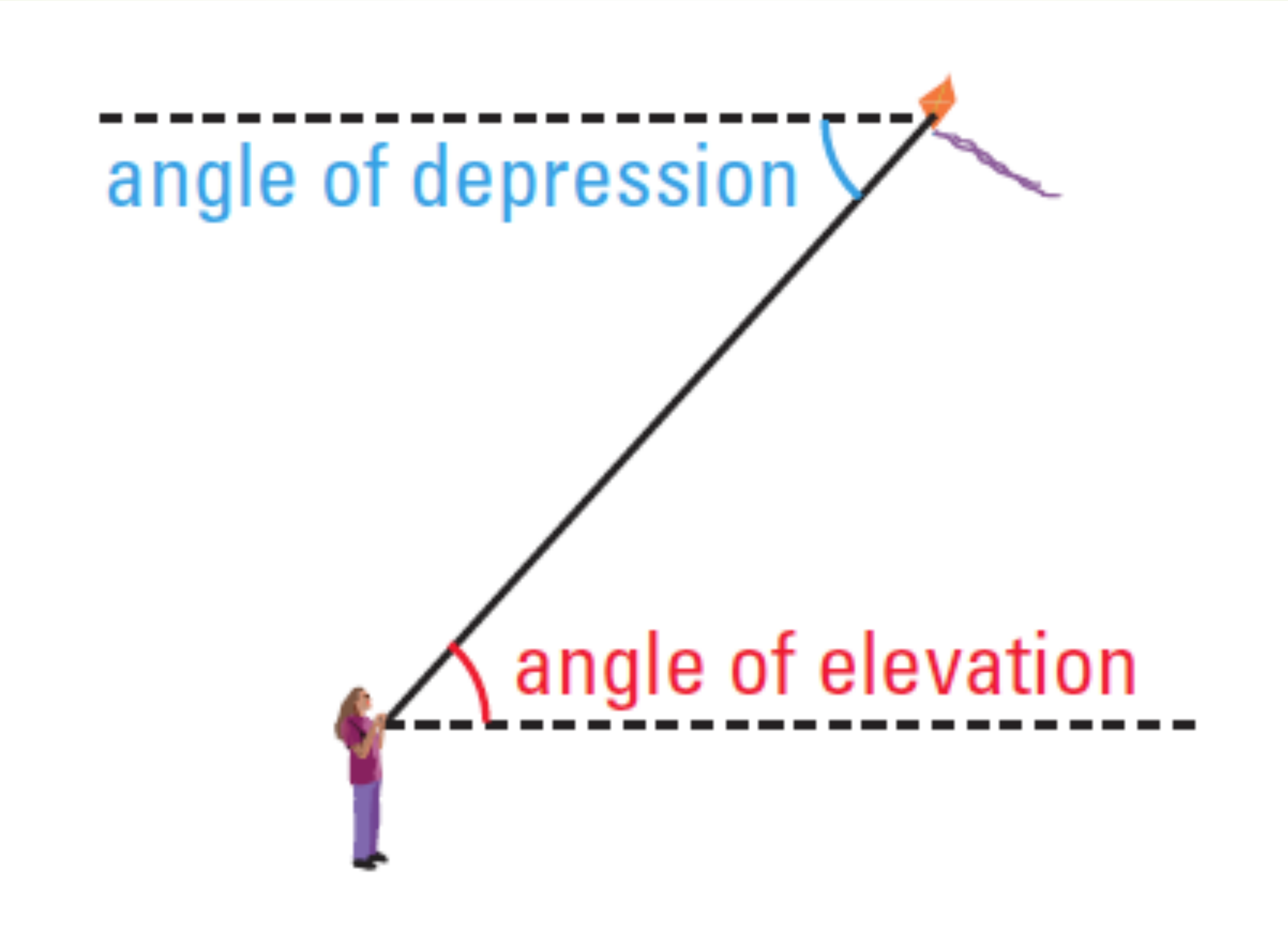
Solving Triangles

- Solving triangles means finding the angles and sides.

Solve $\triangle ABC$.



Using Trigonometry in Real Life



KITE FLYING Wind speed affects the angle at which a kite flies. The table at the right shows the angle the kite line makes with a line parallel to the ground for several different wind speeds. You are flying a kite 4 feet above the ground and are using 500 feet of line. At what altitude is the kite flying if the wind speed is 35 miles per hour?

Wind speed (miles per hour)	Angle of kite line (degrees)
25	70
30	60
35	48
40	29
45	0

12

An airplane flying at an altitude of 30,000 feet is headed toward an airport. To guide the airplane to a safe landing, the airport's landing system sends radar signals from the runway to the airplane at a $10°$ angle of elevation. How far is the airplane (measured along the ground) from the airport runway?

13.2 – General Angles and Radian Measure

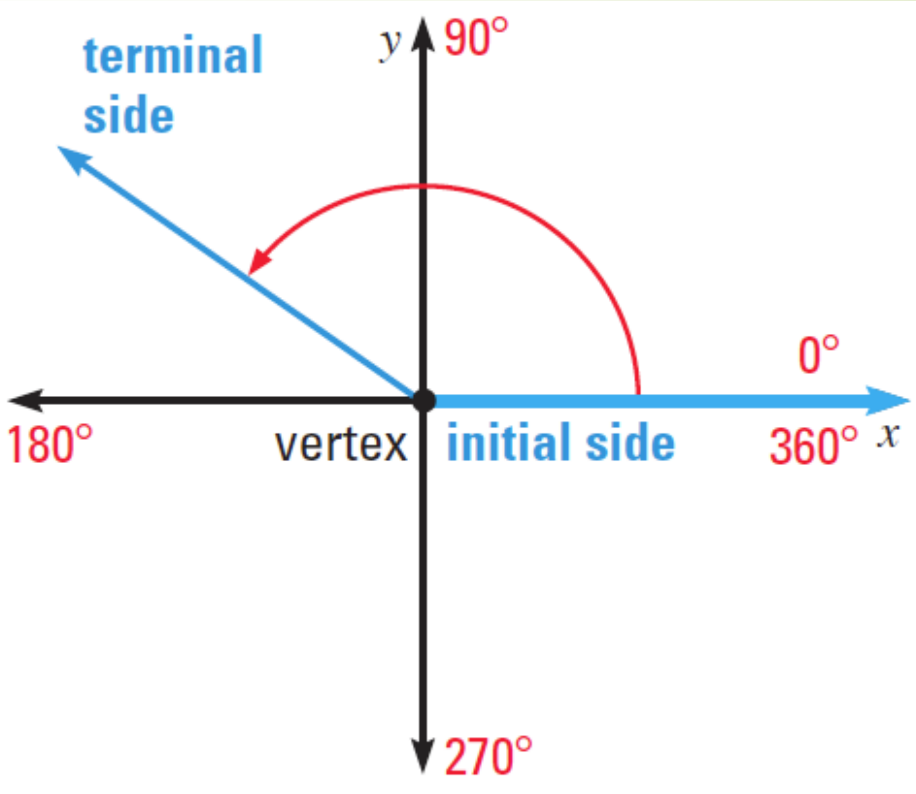
Definitions

Initial side: Fixed side of the angle.

Terminal side: Side that results in the rotation.

Standard position: The initial side is aligned with the x-axis.

Coterminal angles: angles with different measures that have coinciding terminal sides.



Drawing angles in standard position

Draw an angle with the given measure in standard position. Then tell in which quadrant the terminal side lies.

a. $210°$ **b.** $-45°$ **c.** $510°$

Finding coterminal angles

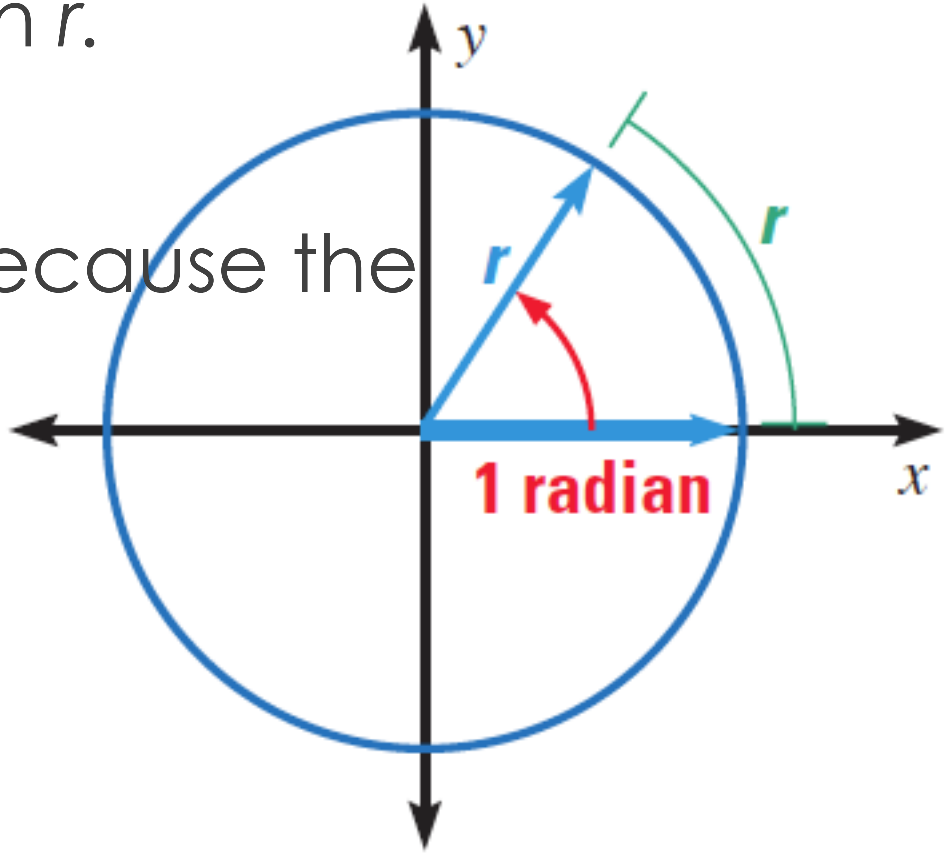
- Coterminal angles can be found by adding or subtracting multiples of 360°.

Find one positive angle and one negative angle that are coterminal with (**a**) $-60°$ and (**b**) 495°.

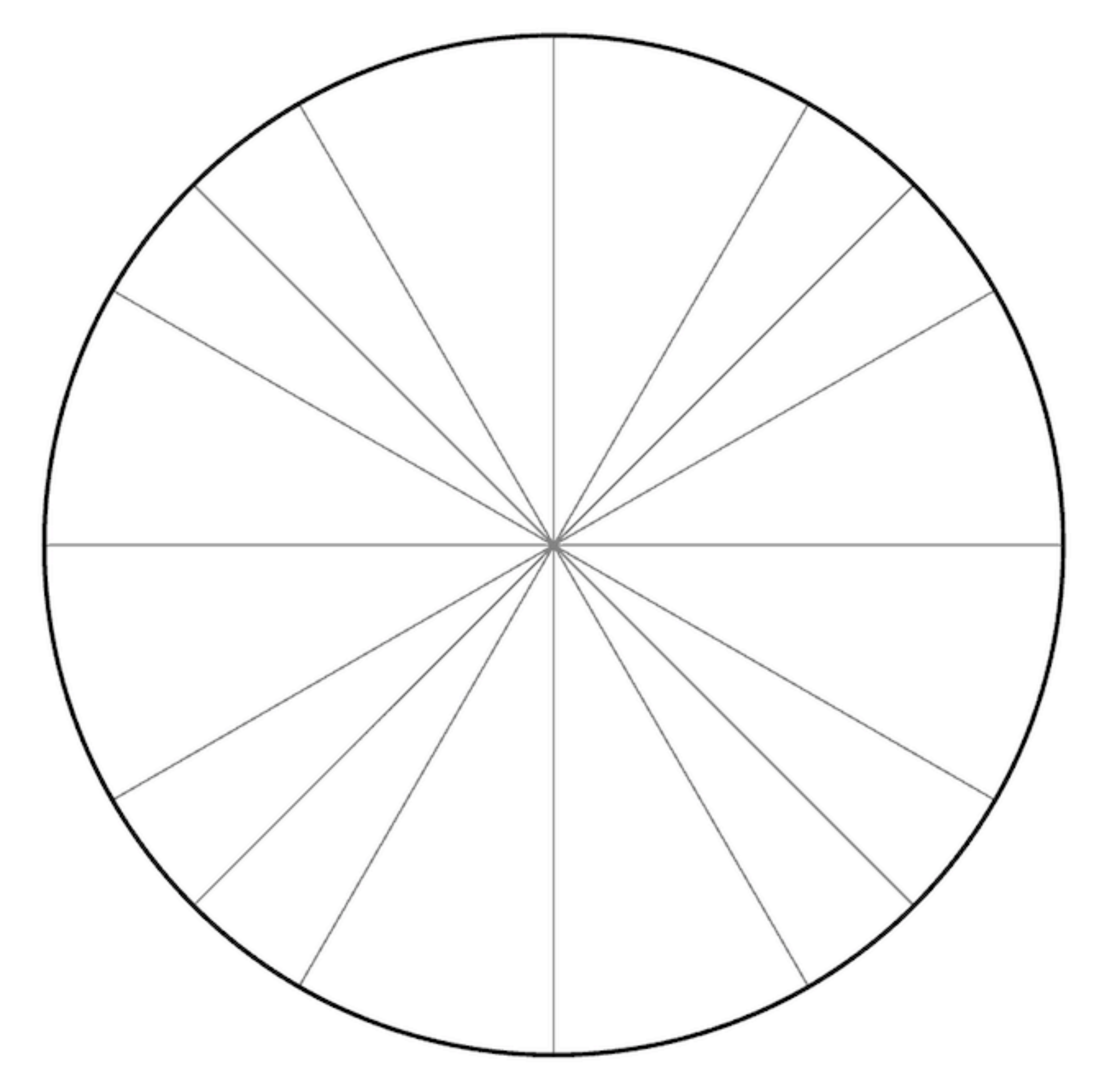
Radians

- In a circle with radius r, one **radian** is the measure of an angle in standard position which intercepts and arc of length *r*.

- There are 2π radians in a circle, because the circumference of a circle is $2\pi r$.



Degrees vs. Radians



Converting between degree and radian

- 1) Write a proportion:

$$\frac{degree\ measure}{180} = \frac{radian\ measure}{\pi}$$

- 2) Solve the proportion.

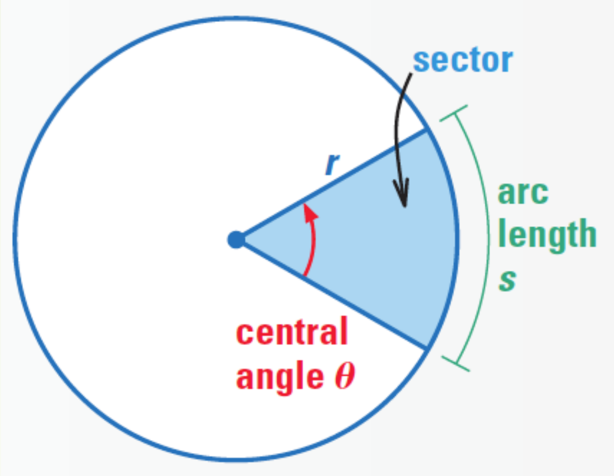
Converting between degree and radian

a. Convert $110°$ to radians.

b. Convert $-\frac{\pi}{9}$ radians to degrees.

Arc length and area of sectors proof

(you will not be tested on this)



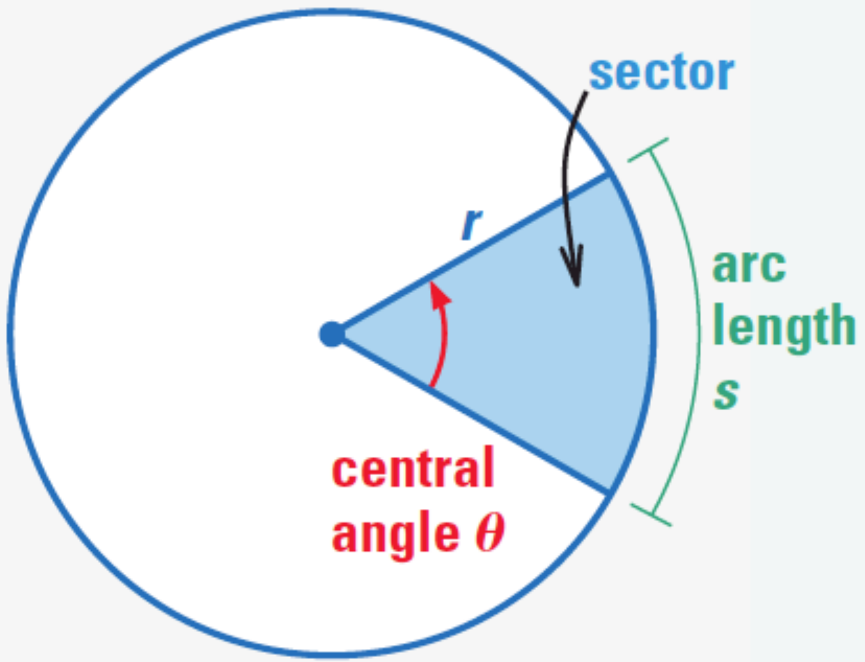
Arc length and area of sectors

ARC LENGTH AND AREA OF A SECTOR

The arc length s and area A of a sector with radius r and central angle θ (measured in radians) are as follows.

Arc length: $s = r\theta$

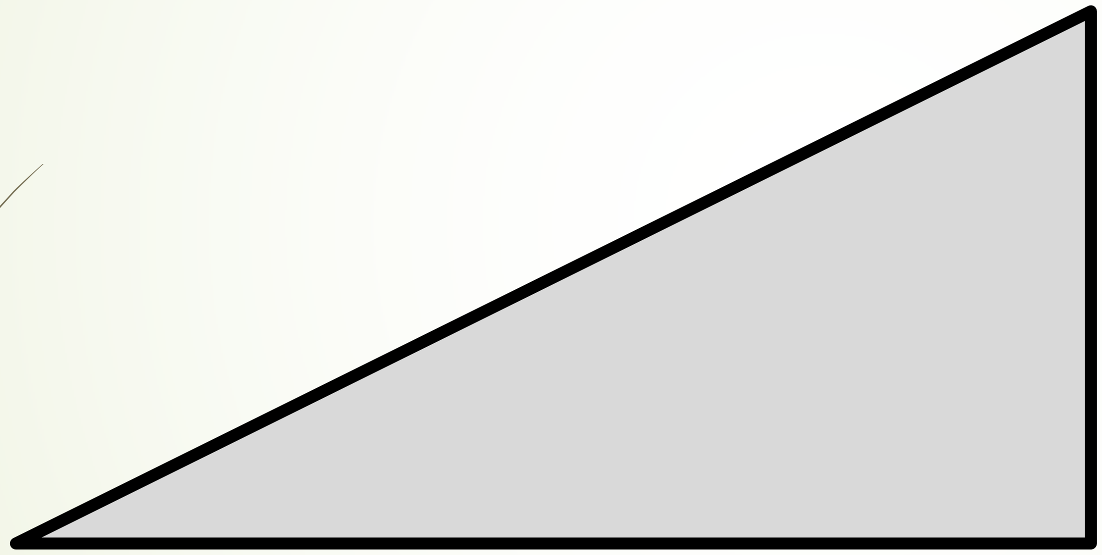
Area: $A = \frac{1}{2}r^2\theta$

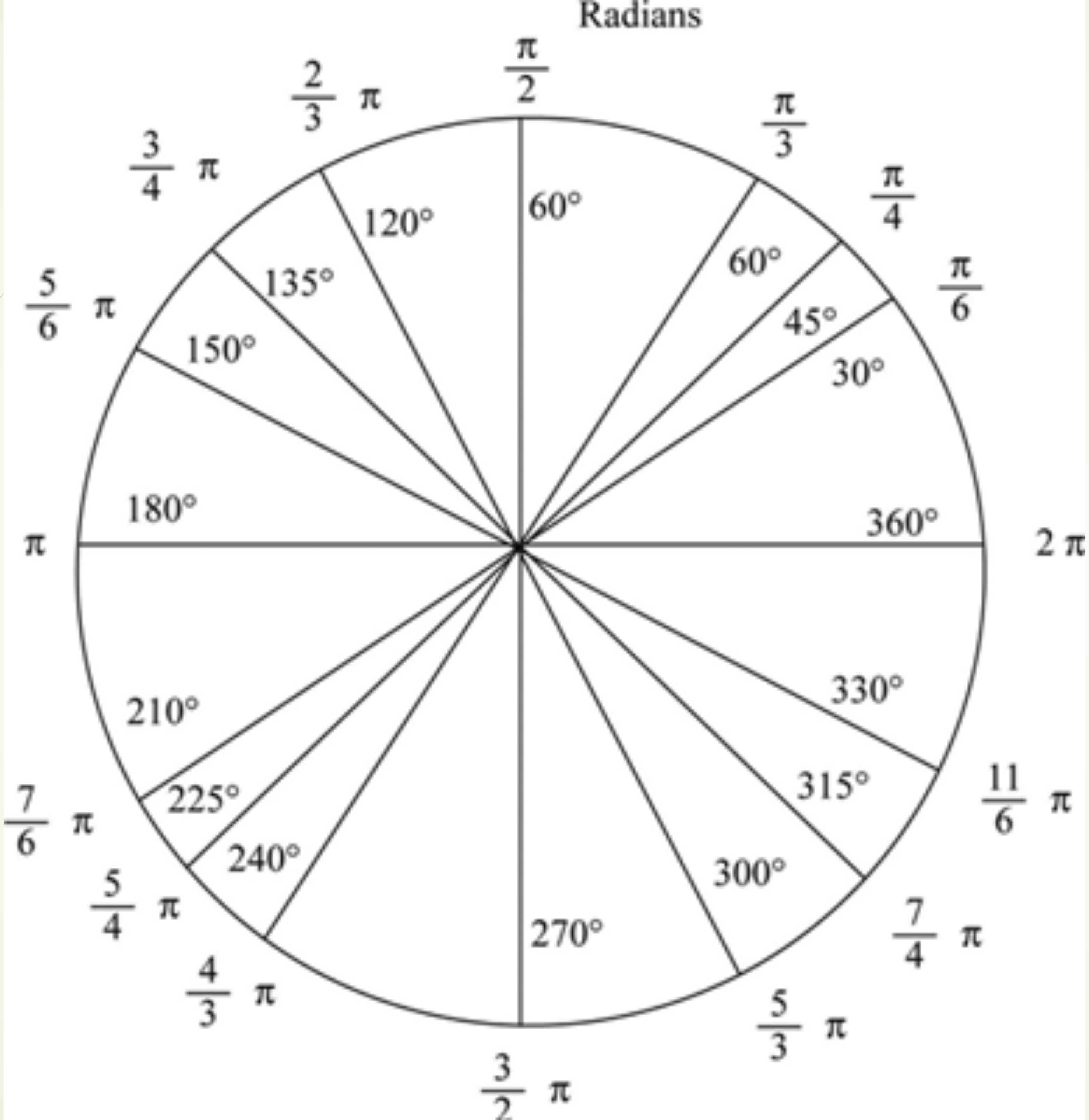


Find the arc length and area of a sector with a radius of 9 cm and a central angle of 60°.

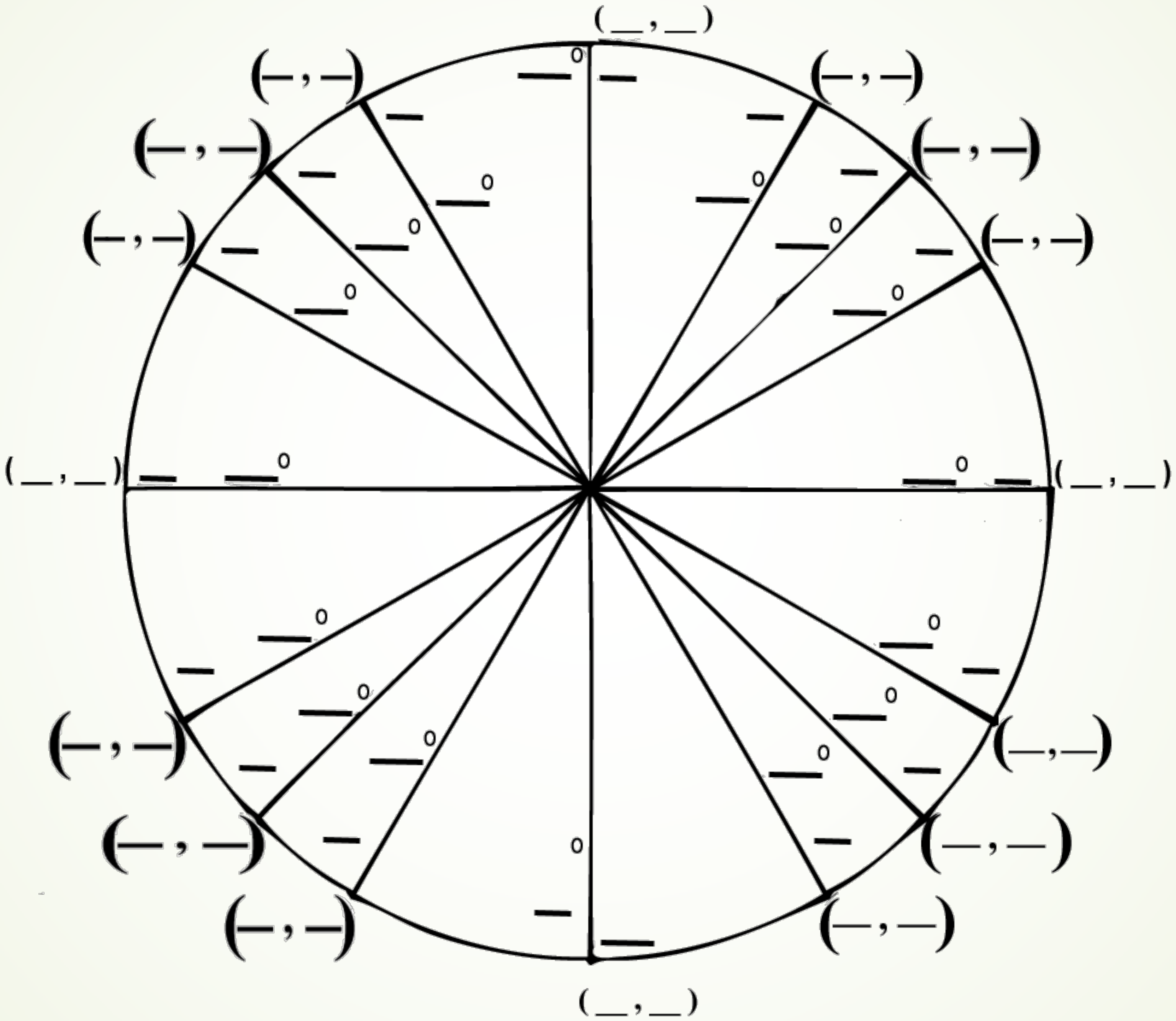
The Unit Circle

Unit right triangles




Radians
π/2
60°
π/3
60°
π/4
45°
π/6
30°
2/3 π
120°
3/4 π
135°
5/6 π
150°
π
180°
360°
2 π
7/6 π
210°
5/4 π
225°
4/3 π
240°
3/2 π
270°
5/3 π
300°
7/4 π
315°
11/6 π
330°

Unit Circle, Fill in the blank



www.mathwarehouse.com/unit-circle

Memorizing the unit circle

- 1) Place your angle on the circle.
- 2) Decide the placement: small x/large y, medium x and y, or large x/small y
- 3) Choose the corresponding sin and cos values: 0, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, 1
- 4) Decide if the values are positive or negative.

Find the sine and cos of each angle:

a) $\frac{\pi}{3}$

b) $\frac{3\pi}{4}$

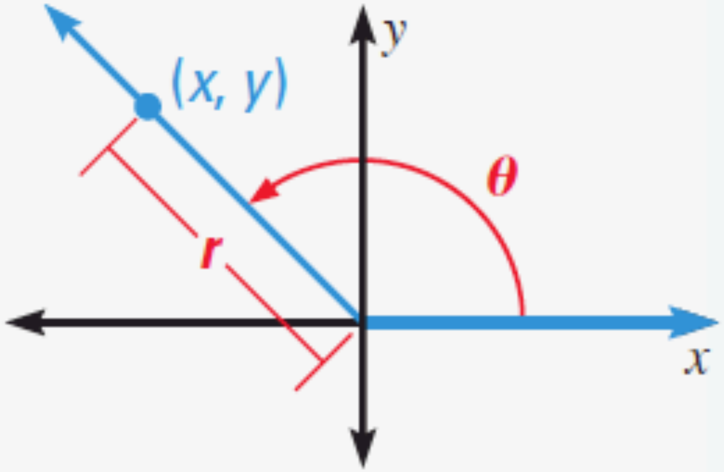
c) $\frac{7\pi}{6}$

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

13.3 – Trigonometric Functions of Any Angle

GENERAL DEFINITION OF TRIGONOMETRIC FUNCTIONS

Let θ be an angle in standard position and (x, y) be any point (except the origin) on the terminal side of θ. The six trigonometric functions of θ are defined as follows.



$\sin \theta = \frac{y}{r}$ $\qquad \csc \theta = \frac{r}{y},\ y \neq 0$

$\cos \theta = \frac{x}{r}$ $\qquad \sec \theta = \frac{r}{x},\ x \neq 0$

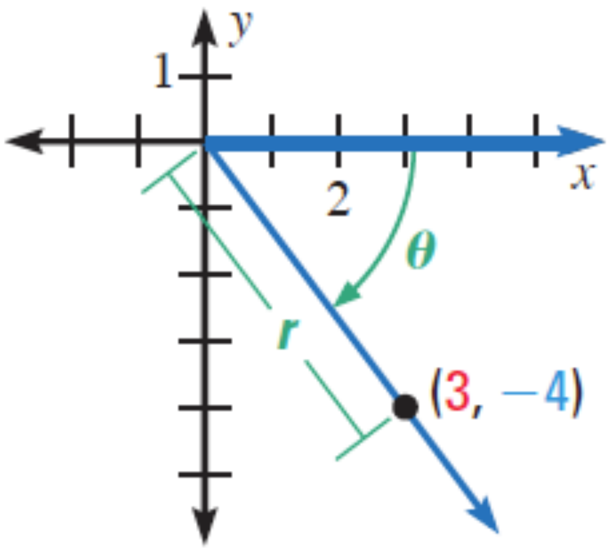
$\tan \theta = \frac{y}{x},\ x \neq 0$ $\qquad \cot \theta = \frac{x}{y},\ y \neq 0$

Pythagorean theorem gives $r = \sqrt{x^2 + y^2}$.

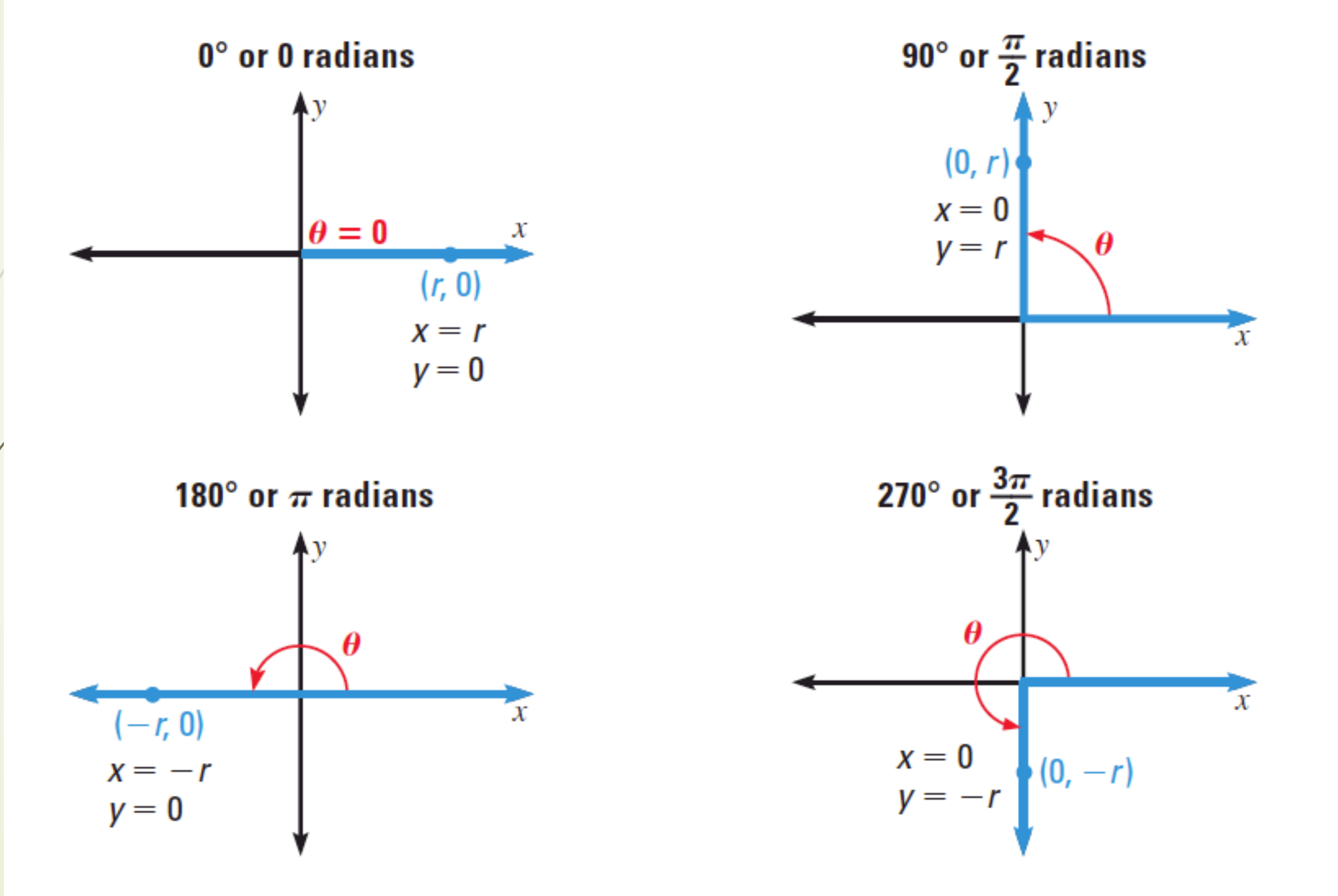
For acute angles, these definitions give the same values as those given by the definitions in Lesson 13.1.

33

Let $(3, -4)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



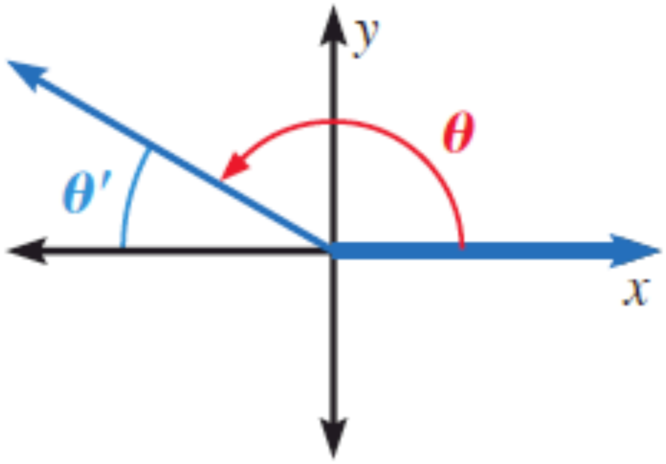
Quadrangles



Reference Angles

$90° < \theta < 180°;$

$\frac{\pi}{2} < \theta < \pi$

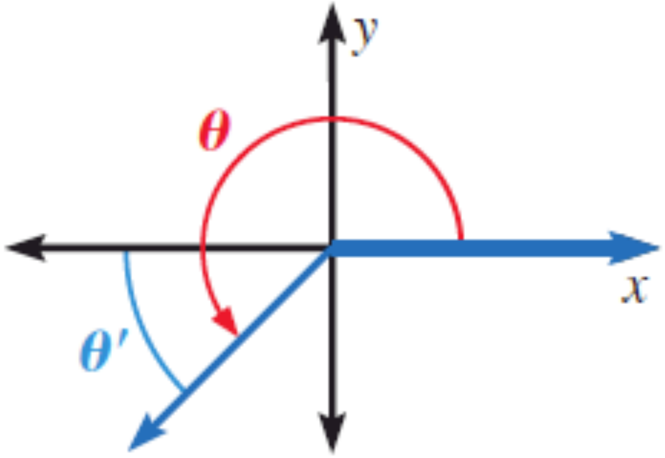


Degrees: $\theta' = 180° - \theta$

Radians: $\theta' = \pi - \theta$

$180° < \theta < 270°;$

$\pi < \theta < \frac{3\pi}{2}$

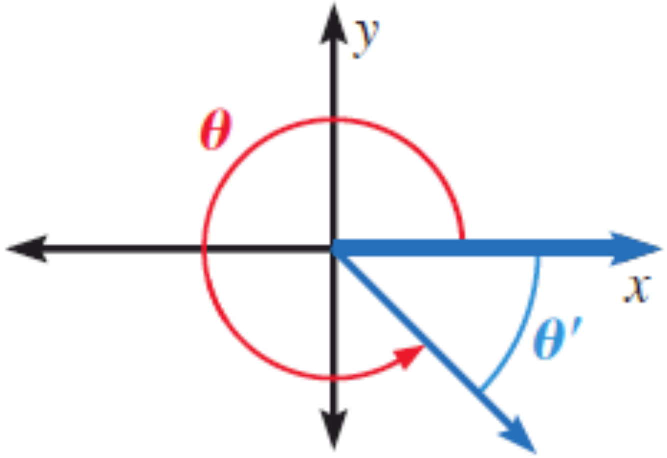


Degrees: $\theta' = \theta - 180°$

Radians: $\theta' = \theta - \pi$

$270° < \theta < 360°;$

$\frac{3\pi}{2} < \theta < 2\pi$



Degrees: $\theta' = 360° - \theta$

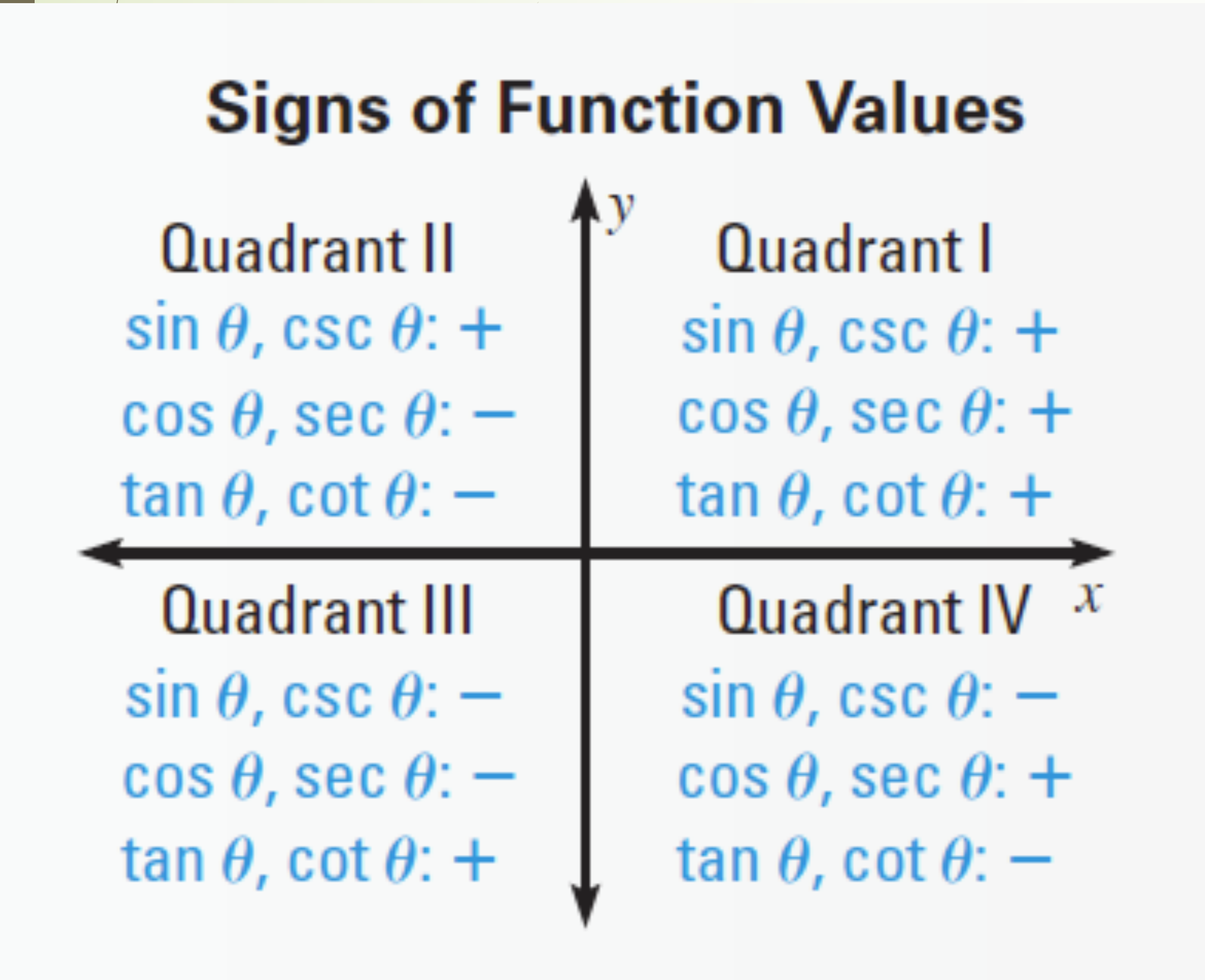
Radians: $\theta' = 2\pi - \theta$

Finding reference angles

Find the reference angle θ' for each angle θ.

a. $\theta = 320°$ **b.** $\theta = -\frac{5\pi}{6}$

Sign of trigonometric functions



Evaluating Trigonometric Functions

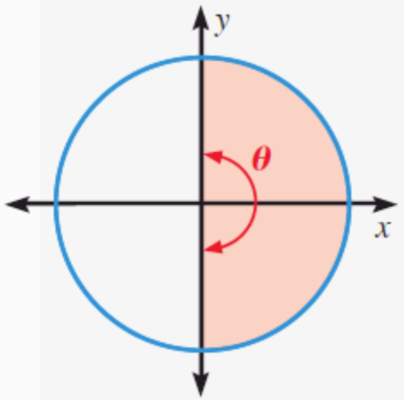
- 1) Find the reference angles.
- 2) Evaluate the trigonometric function for the reference angle.
- 3) Decide the sign of the trigonometric function for the angle based on its quadrant.

Evaluate (**a**) $\tan(-210^\circ)$ and (**b**) $\csc \frac{11\pi}{4}$.

13.4 – Inverse Trigonometry Functions

INVERSE TRIGONOMETRIC FUNCTIONS

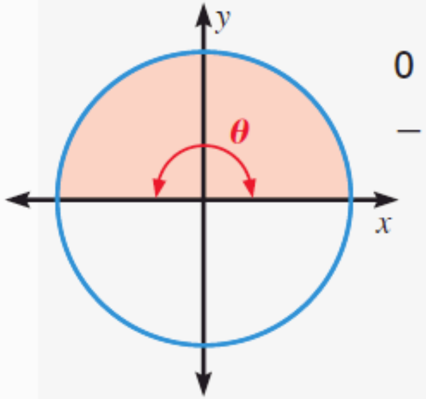
- If $-1 \le a \le 1$, then the **inverse sine** of a is $\sin^{-1} a = \theta$ where $\sin\theta = a$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (or $-90° \le \theta \le 90°$).



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$-1 \le \sin\theta \le 1$

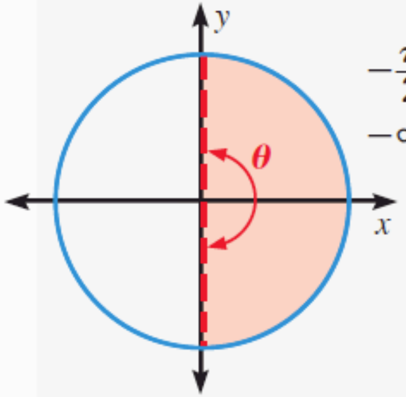
- If $-1 \le a \le 1$, then the **inverse cosine** of a is $\cos^{-1} a = \theta$ where $\cos\theta = a$ and $0 \le \theta \le \pi$ (or $0° \le \theta \le 180°$).



$0 \le \theta \le \pi$

$-1 \le \cos\theta \le 1$

- If a is any real number, then the **inverse tangent** of a is $\tan^{-1} a = \theta$ where $\tan\theta = a$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (or $-90° < \theta < 90°$).



$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$-\infty < \tan\theta < +\infty$

Evaluating Inverse Trigonometric Functions

Evaluate the expression in both radians and degrees.

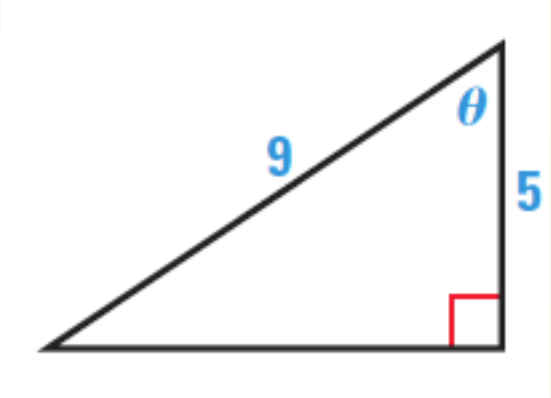
a. $\sin^{-1} \frac{\sqrt{3}}{2}$ **b.** $\cos^{-1} 2$ **c.** $\tan^{-1}(-1)$

Solving a Trigonometric Equation

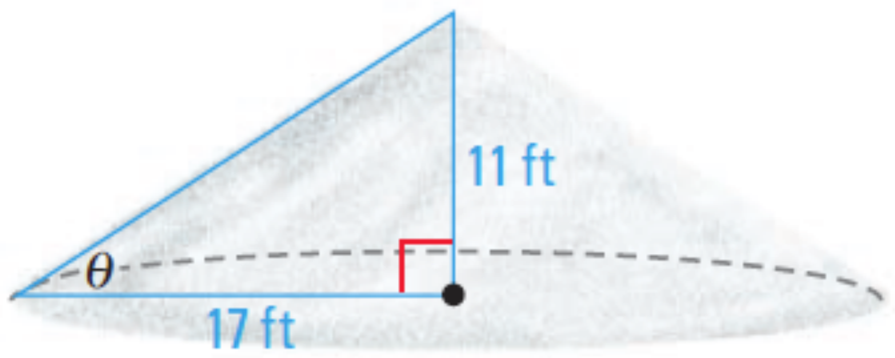
Solve the equation $\sin\theta = -\frac{1}{4}$ where $180° < \theta < 270°$.

Finding Angle Measures

Find the measure of the angle θ for the triangle shown.



ROCK SALT Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose*. When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. ▶ Source: Bulk-Store Structures, Inc.



a. Find the angle of repose for rock salt.

b. How tall is a pile of rock salt that has a base diameter of 50 feet?

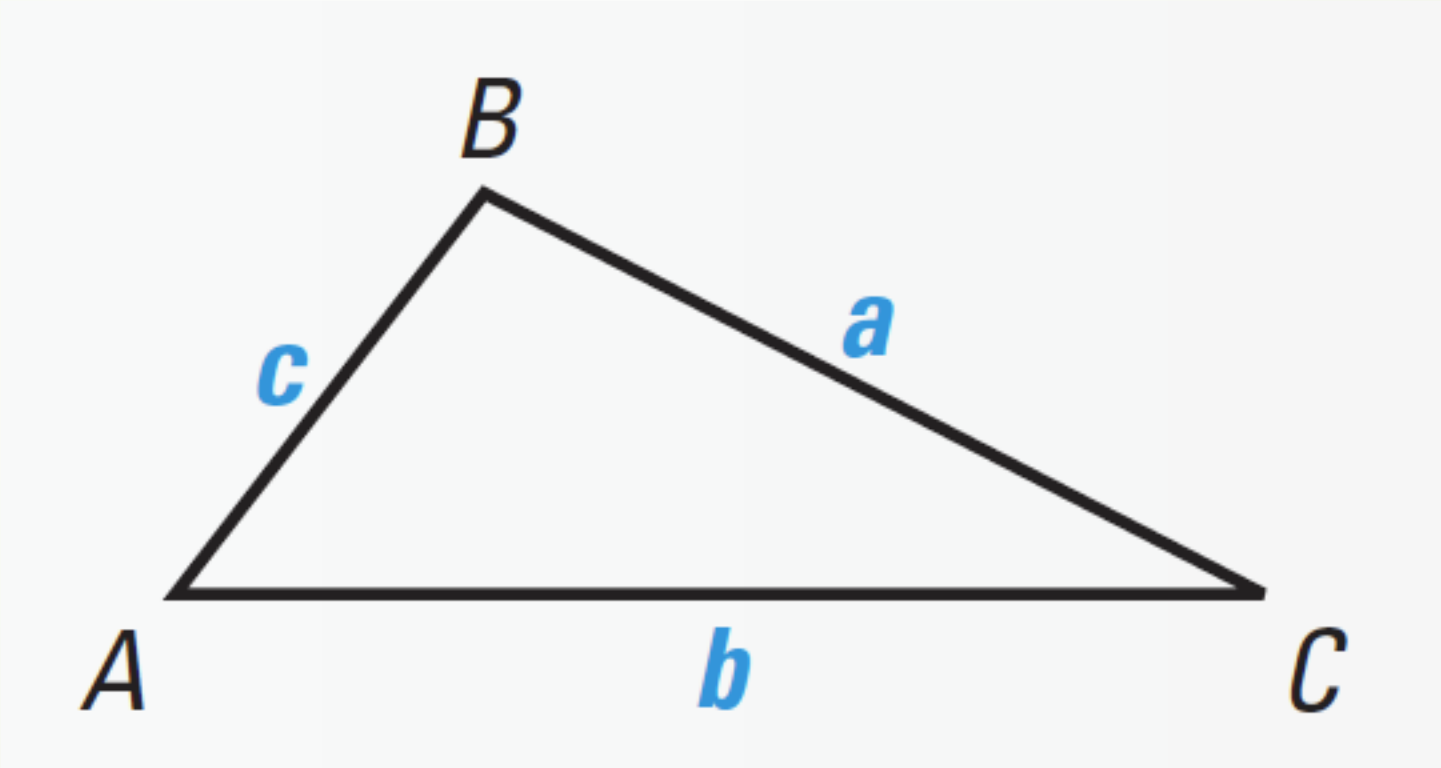
13.5 – The Law of Sines

Cases of non-right triangles to solve

- **1) Two angles and any side (AAS or ASA)**
- **2) Two sides and an opposite angle (SSA)**

- 3) Three sides (SSS)
- 4) Two sides and their included angles (SAS)

Labelling Triangles



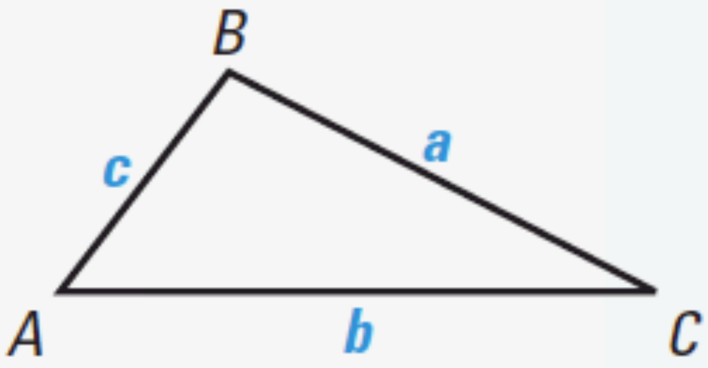
The law of sines can be used when you have an angle and opposite side, and one more angle or side.

LAW OF SINES

If $\triangle ABC$ has sides of length a, b, and c as shown, then:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

An equivalent form is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.



AAS or ASA Case - only one triangle determined

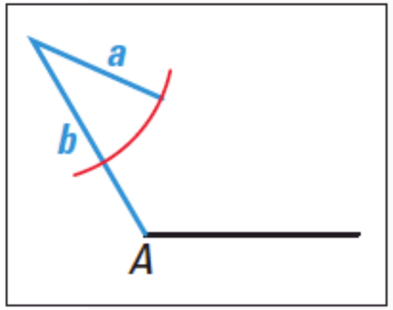
Solve $\triangle ABC$ with $C = 103°$, $B = 28°$, and $b = 26$ feet.

SSA possible cases

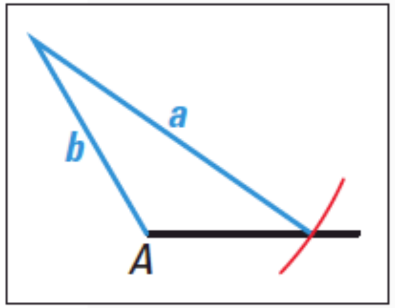
POSSIBLE TRIANGLES IN THE SSA CASE

Consider a triangle in which you are given *a, b,* and *A*.

A IS OBTUSE.

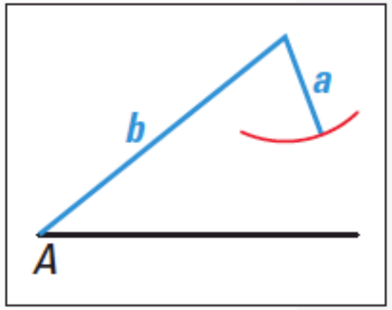


$a \leq b$
No triangle

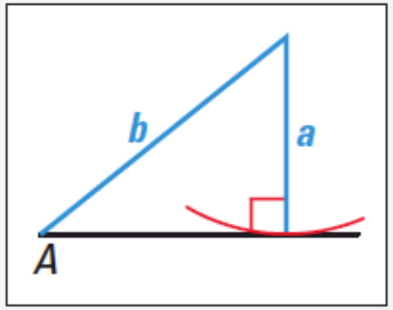


$a > b$
One triangle

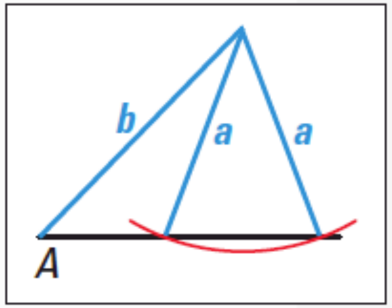
A IS ACUTE.



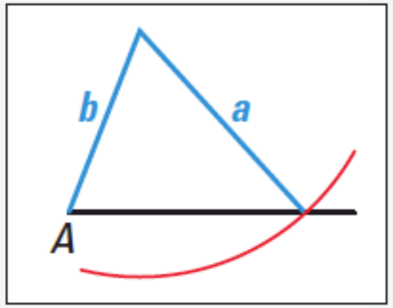
$b \sin A > a$
No triangle



$b \sin A = a$
One triangle



$b \sin A < a < b$
Two triangles



$a > b$
One triangle

SSA – One Triangle Case

Solve $\triangle ABC$ with $C = 122°$, $a = 12$ cm, and $c = 18$ cm.

SSA – No Triangle Case

Solve $\triangle ABC$ with $a = 4$ inches, $b = 2.5$ inches, and $B = 58°$.

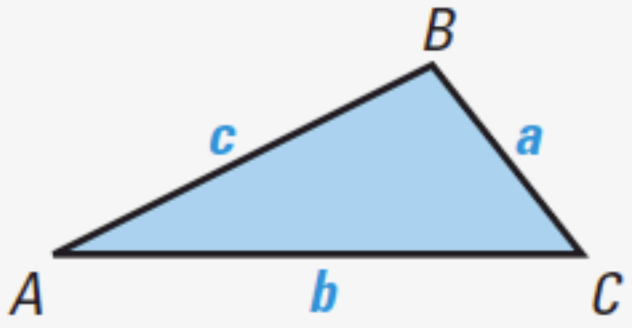
SSA – Two Triangles Case

Solve $\triangle ABC$ with $B = 56°$, $b = 13$, $a = 14$

Area of a triangle

AREA OF A TRIANGLE

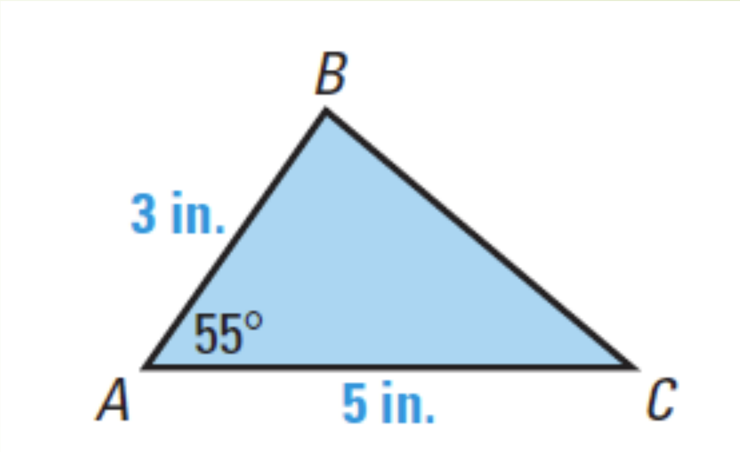
The area of any triangle is given by one half the product of the lengths of two sides times the sine of their included angle. For $\triangle ABC$ shown, there are three ways to calculate the area:



$$\text{Area} = \frac{1}{2}bc \sin A \qquad \text{Area} = \frac{1}{2}ac \sin B \qquad \text{Area} = \frac{1}{2}ab \sin C$$

57

Find the area of $\triangle ABC$.



13.6 – The Law of Cosines

Cases of non-right triangles to solve

- 1) Two angles and any side (AAS or ASA)
- 2) Two sides and an opposite angle (SSA)

- **3) Three sides (SSS)**
- **4) Two sides and their included angles (SAS)**

The law of cosines should be used when the law of sines doesn't work.

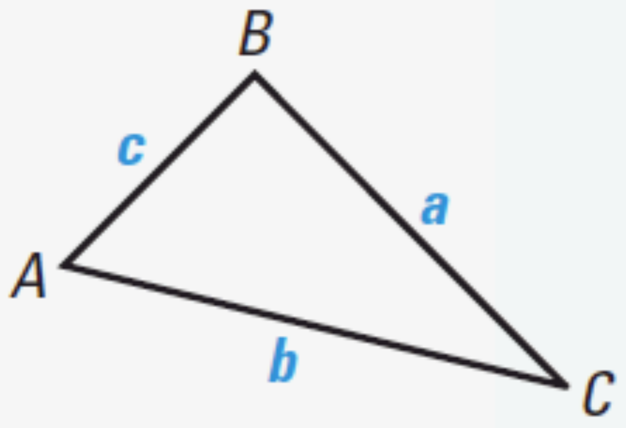
LAW OF COSINES

If $\triangle ABC$ has sides of length a, b, and c as shown, then:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

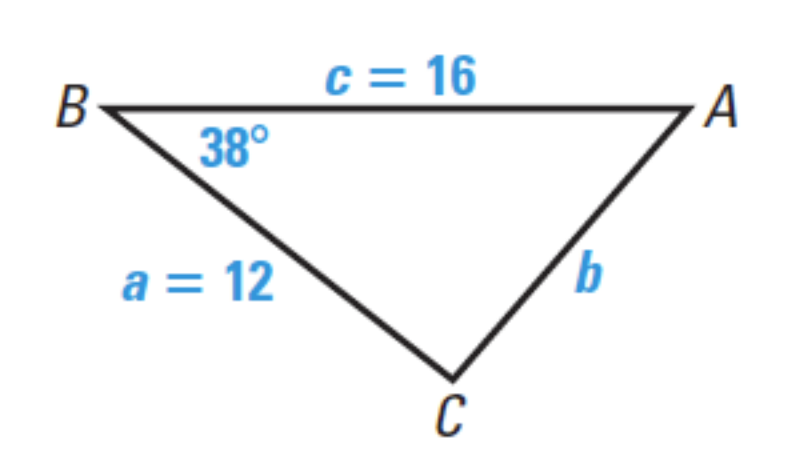
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



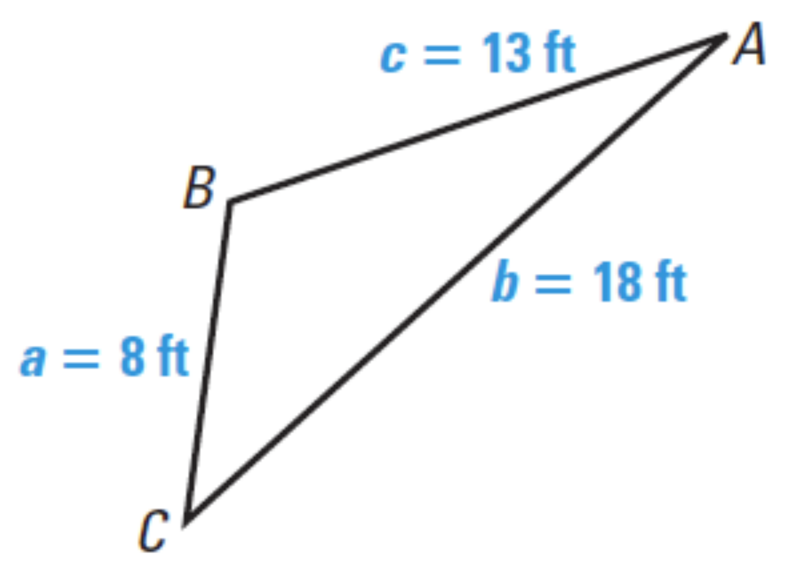
The SAS case

Solve $\triangle ABC$ with $a = 12$, $c = 16$, and $B = 38°$.



The SSS case

Solve $\triangle ABC$ with $a = 8$ feet, $b = 18$ feet, and $c = 13$ feet.



Area of a Triangle – Heron's formula

HERON'S AREA FORMULA

The area of the triangle with sides of length *a*, *b*, and *c* is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$. The variable *s* is called the *semiperimeter*, or half-perimeter, of the triangle.

64

Find the area of $\triangle ABC$.