CHAPTER 8

Exponential and Logarithmic functions

8.1/8.2 - EXPONENTIAL GROWTH AND DECAY

Basic exponential function

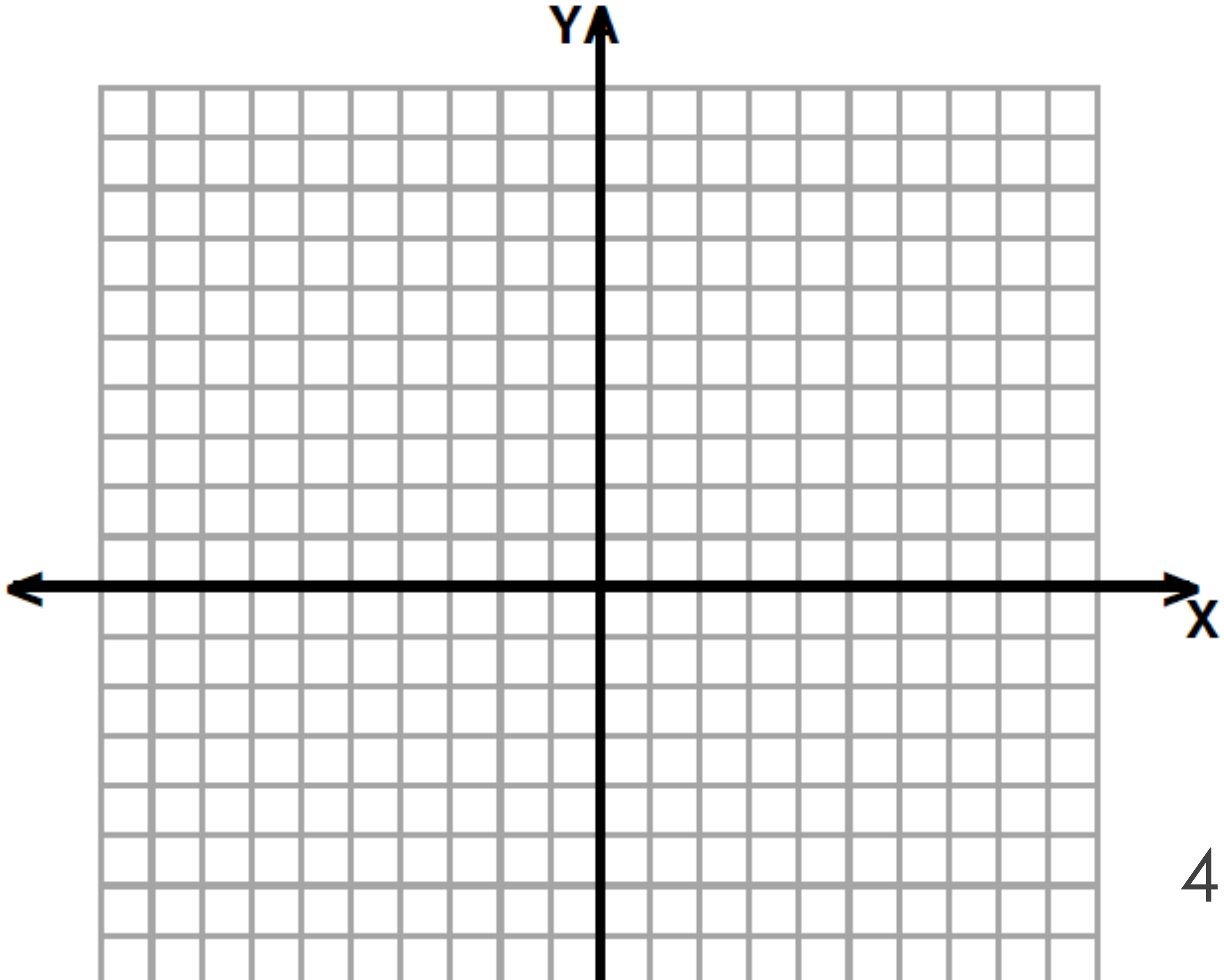
$$f(x) = b^x$$

- It has a horizontal **asymptote** because it will never reach zero, no matter what x-value you put in the function.

Growth vs. Decay

Graph $f(x) = 2^x$

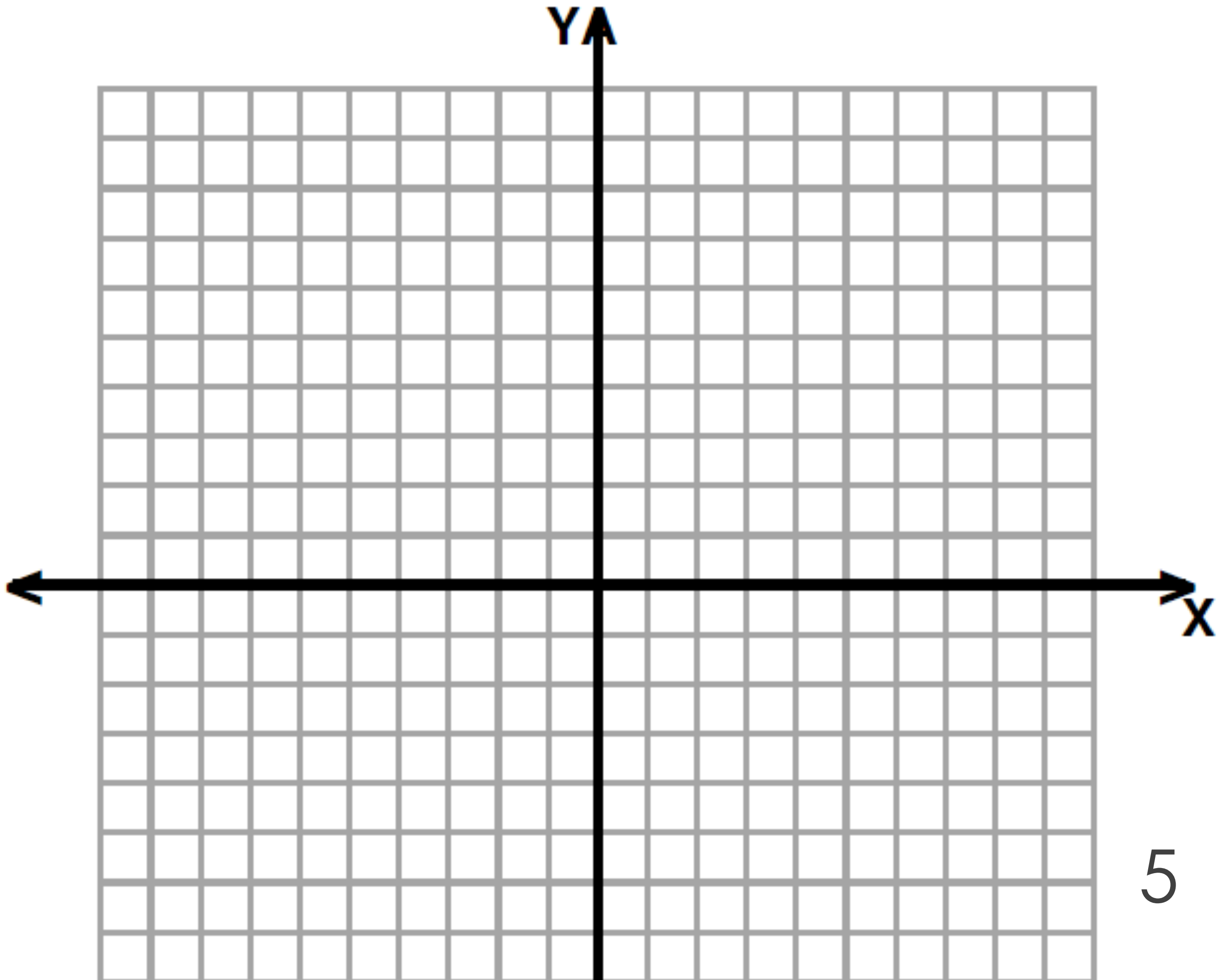
x	$f(x) = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	



Growth vs. Decay

Graph $f(x) = \left(\frac{1}{2}\right)^x$

x	$f(x) = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	

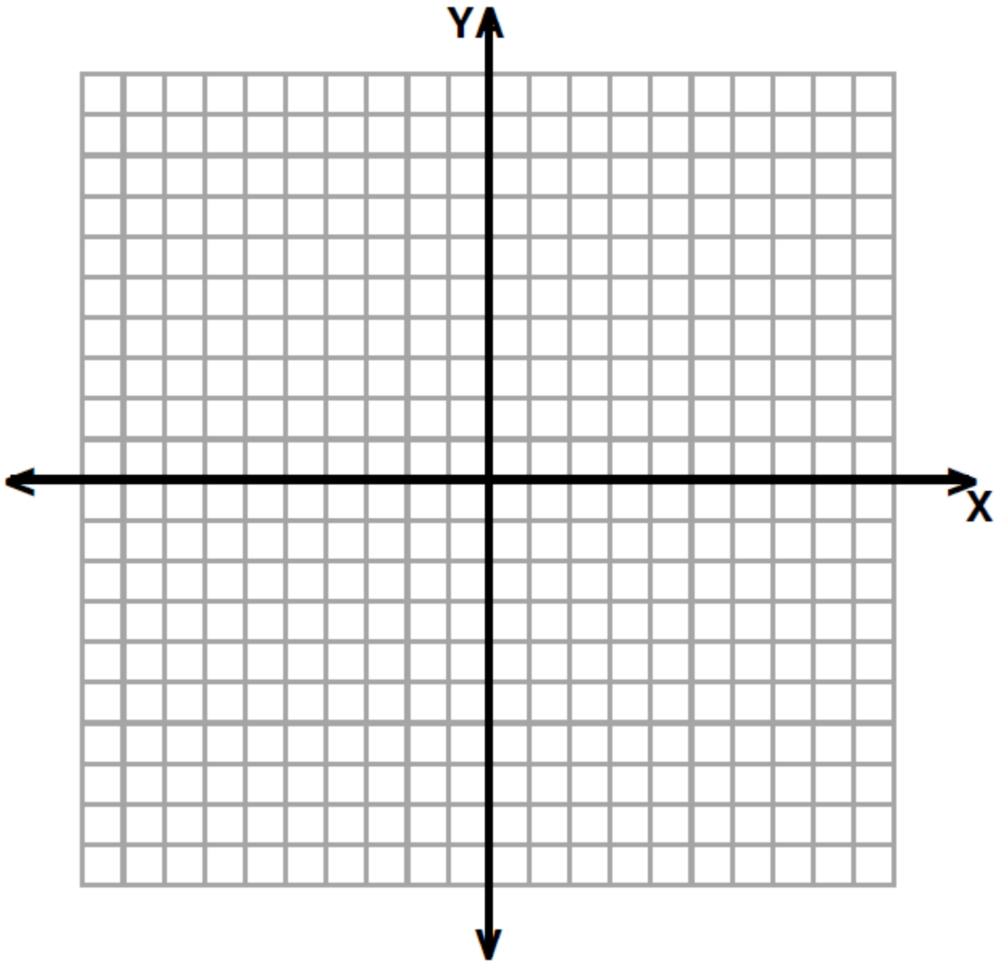


Growth vs. Decay

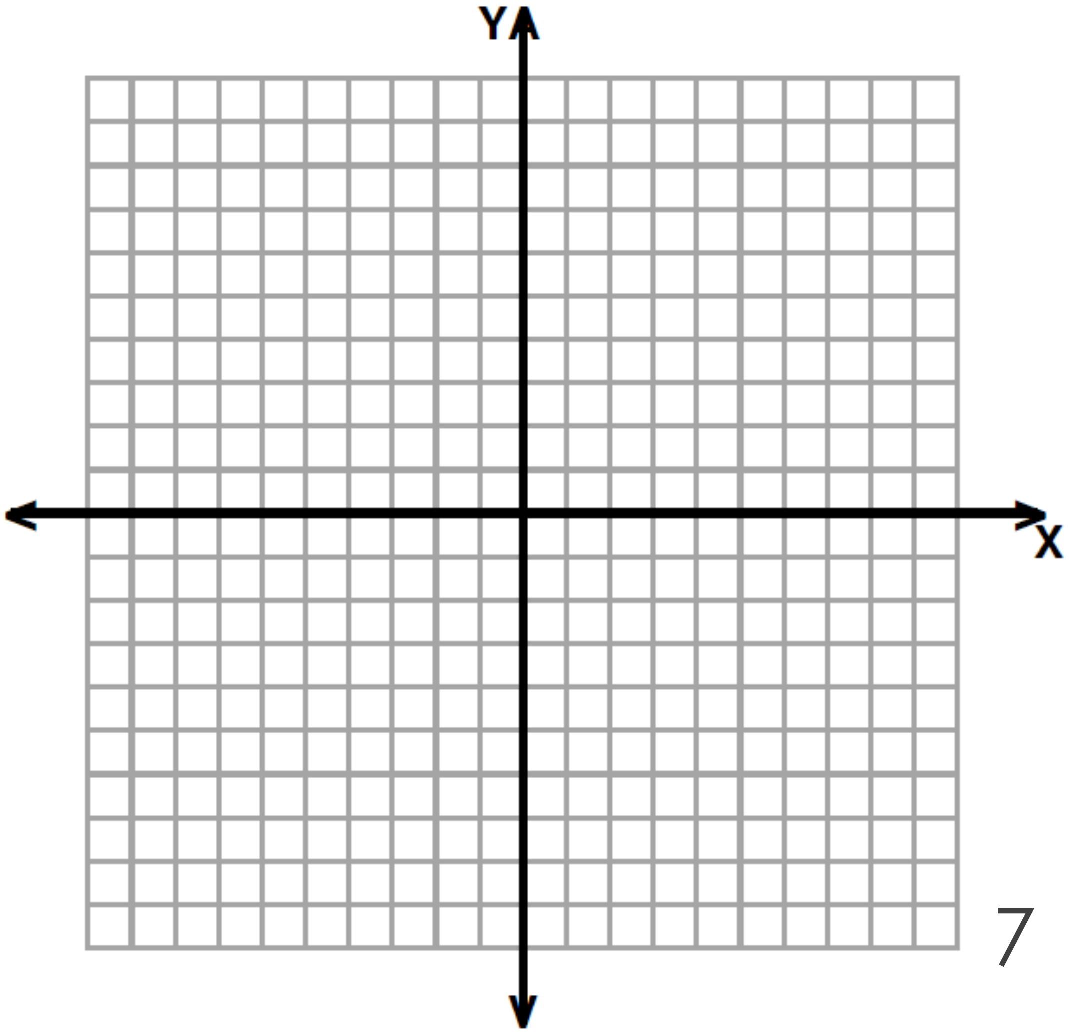
Growth	Decay
• •	• •

Parameter "a" $f(x) = a \cdot b^x$

a. $y = 3\left(\frac{1}{4}\right)^x$



b. $y = -\left(\frac{3}{2}\right)^x$



Steps for graphing $f(x) = a \cdot b^{(x-h)} + k$

1) Create a table of values for the basic functions, using -1,0, and 1 for x.

x	-1	0	1
y	1/b	1	b

2) Apply the transformations using a, h, and k to get the actual points and create a new table of values.

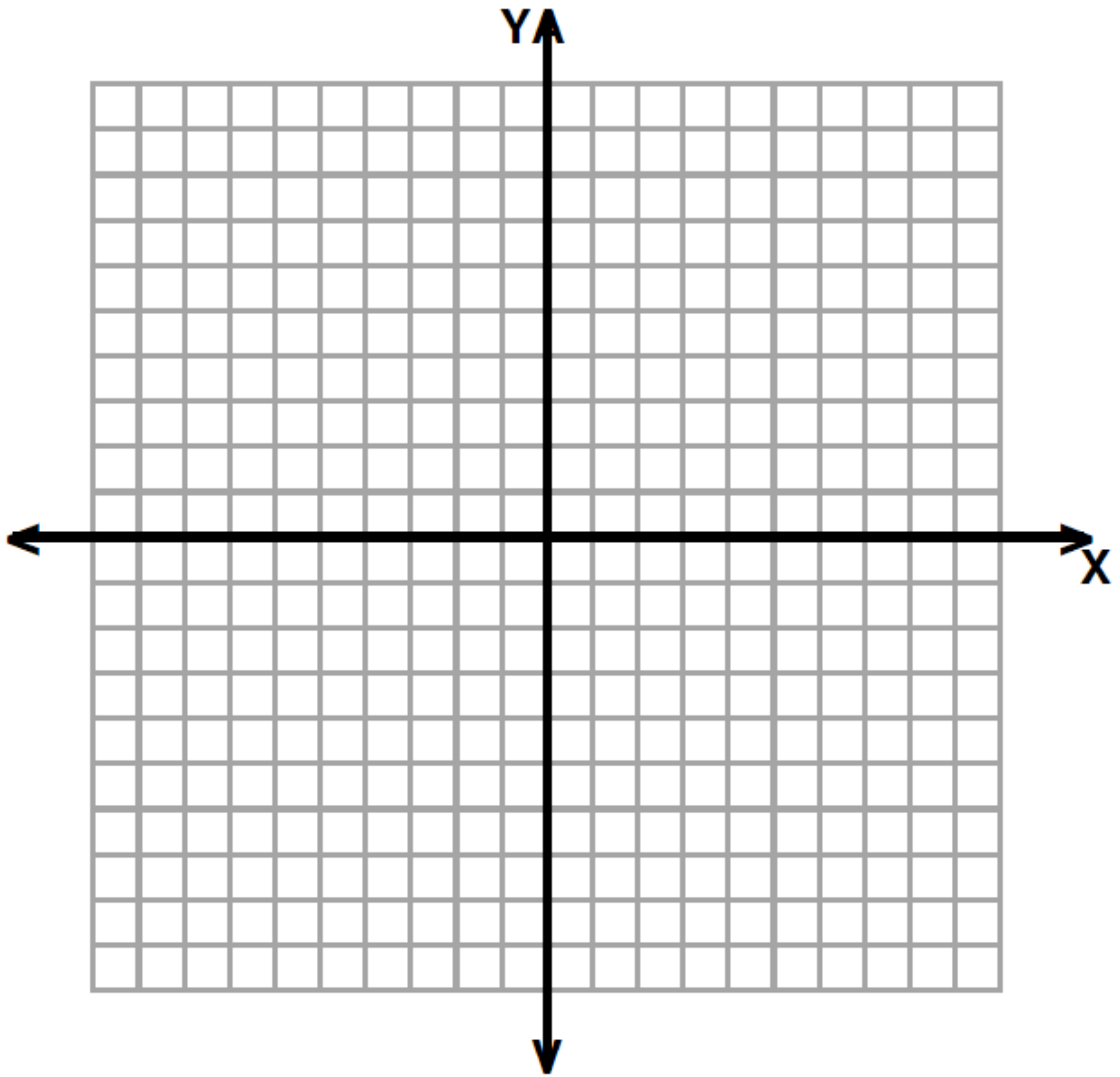
x→ x+h

y→ ay+k

3) Draw the asymptote y=k

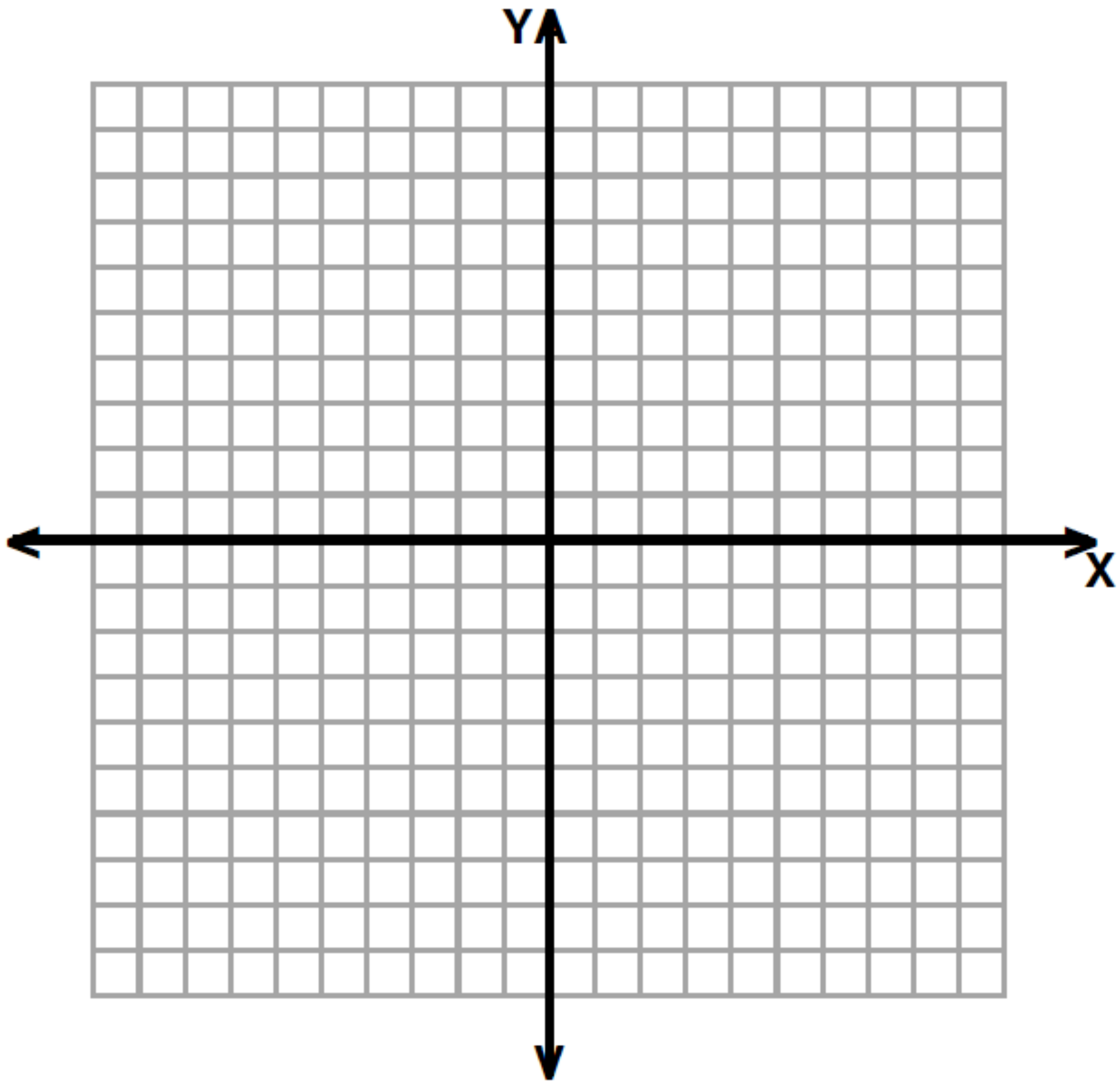
Parameters "h" and "k" $f(x) = a \cdot b^{(x-h)} + k$

$y = 3 \cdot 2^{x-1} - 4$



Parameters "h" and "k" $f(x) = a \cdot b^{(x-h)} + k$

$y = -3\left(\frac{1}{2}\right)^{x+2} + 1$



Exponentials to represent growth or decay.

When a real-life quantity increases by a fixed percent each year, the quantity can be modeled by:

$$y = a(1 + r)^t$$

For decay: $y = a(1 - r)^t$

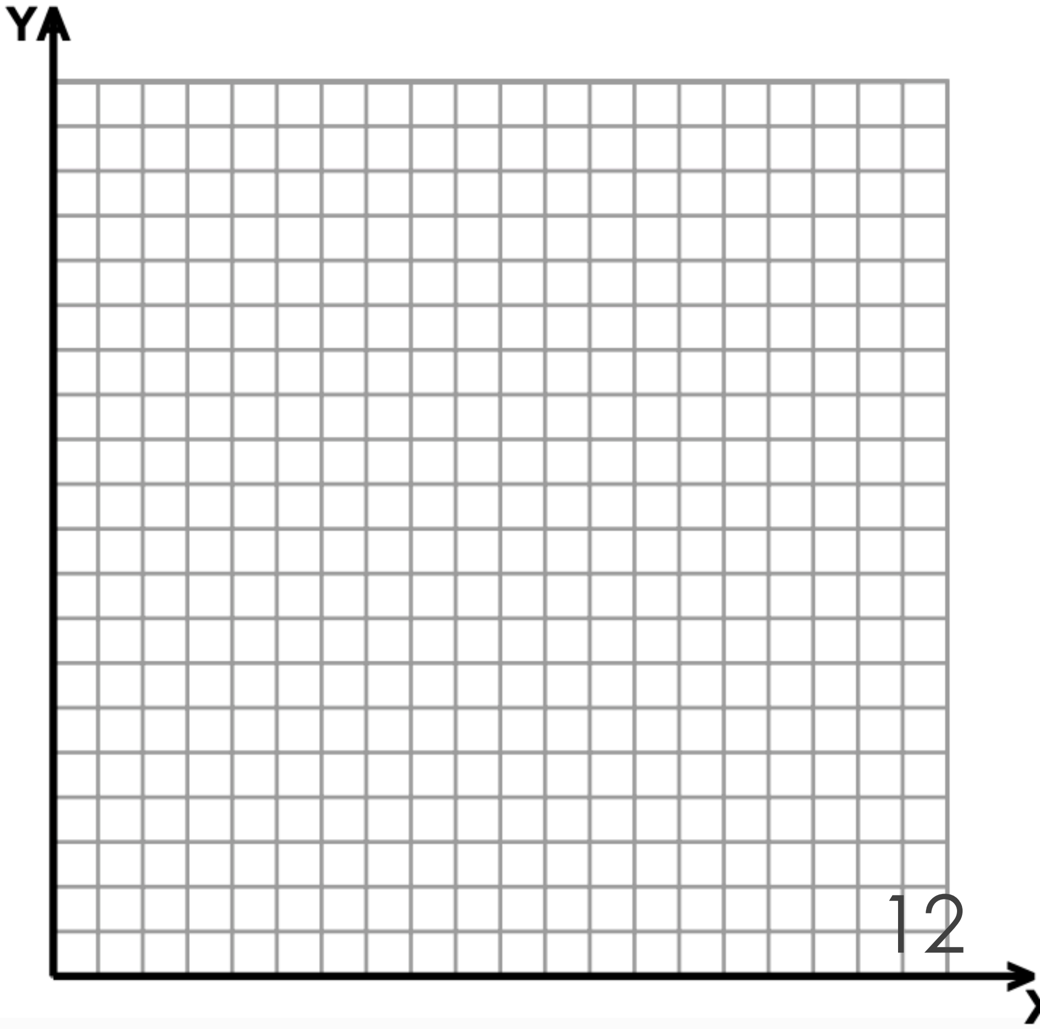
INTERNET HOSTS In January, 1993, there were about 1,313,000 Internet hosts. During the next five years, the number of hosts increased by about 100% per year.

▶ Source: Network Wizards

a. Write a model giving the number h (in millions) of hosts t years after 1993. About how many hosts were there in 1996?

b. Graph the model.

c. Use the graph to estimate the year when there were 30 million hosts.

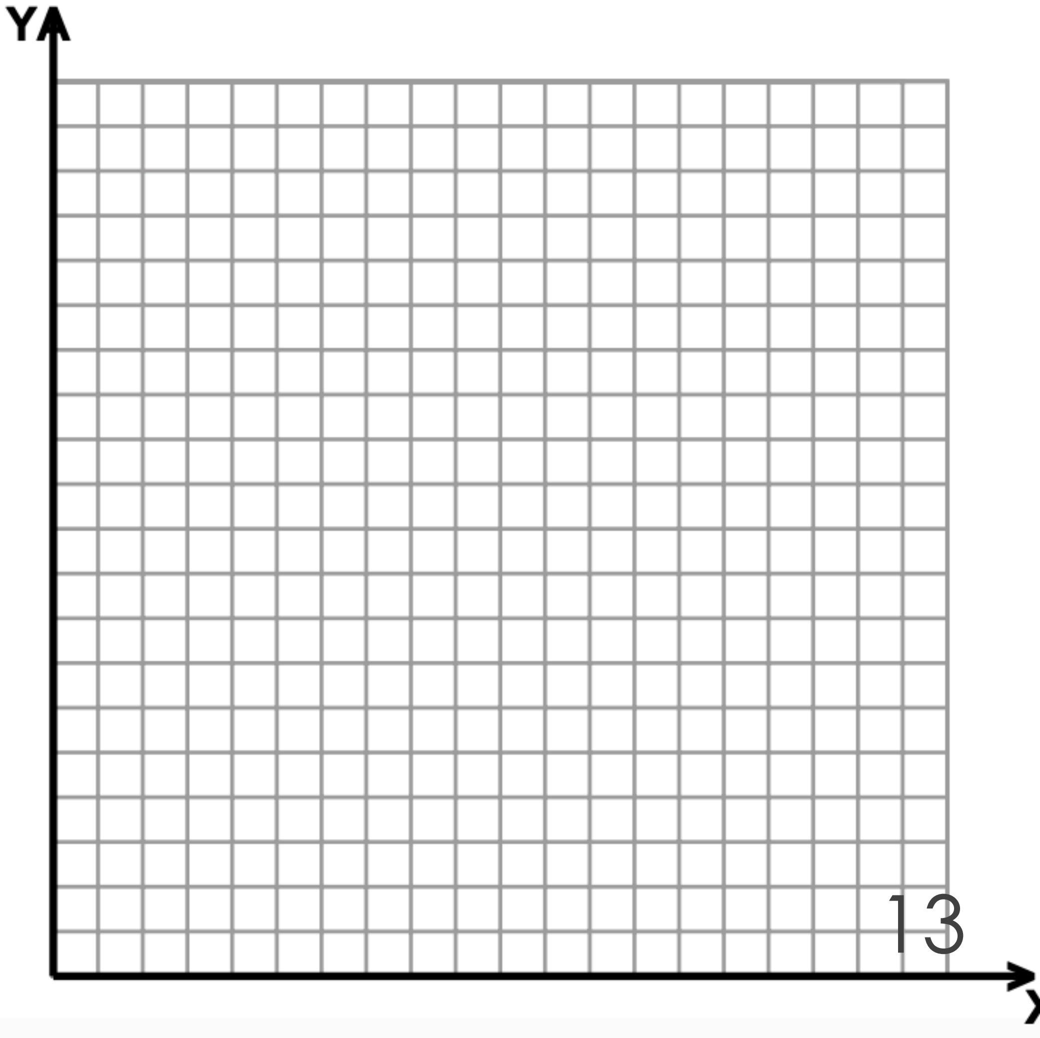


You buy a new car for \$24,000. The value y of the car decreases by 16% each year.

a. Write an exponential decay model for the value of the car. Use the model to estimate the value after 2 years.

b. Graph the model.

c. Use the graph to estimate when the car will have a value of \$12,000.



Compound interest

COMPOUND INTEREST

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years can be modeled by this equation:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

FINANCE You deposit $1000 in an account that pays 8% annual interest. Find the balance after 1 year if the interest is compounded with the given frequency.

a. annually **b.** quarterly **c.** daily

8.3 – THE NUMBER e

ACTIVITY

Developing Concepts

Investigating the Natural Base *e*

1 Copy the table and use a calculator to complete the table.

n	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1+\frac{1}{n}\right)^n$	2.594	?	?	?	?	?

2 Do the values in the table appear to be approaching a fixed decimal number? If so, what is the number rounded to three decimal places?

THE NATURAL BASE *e*

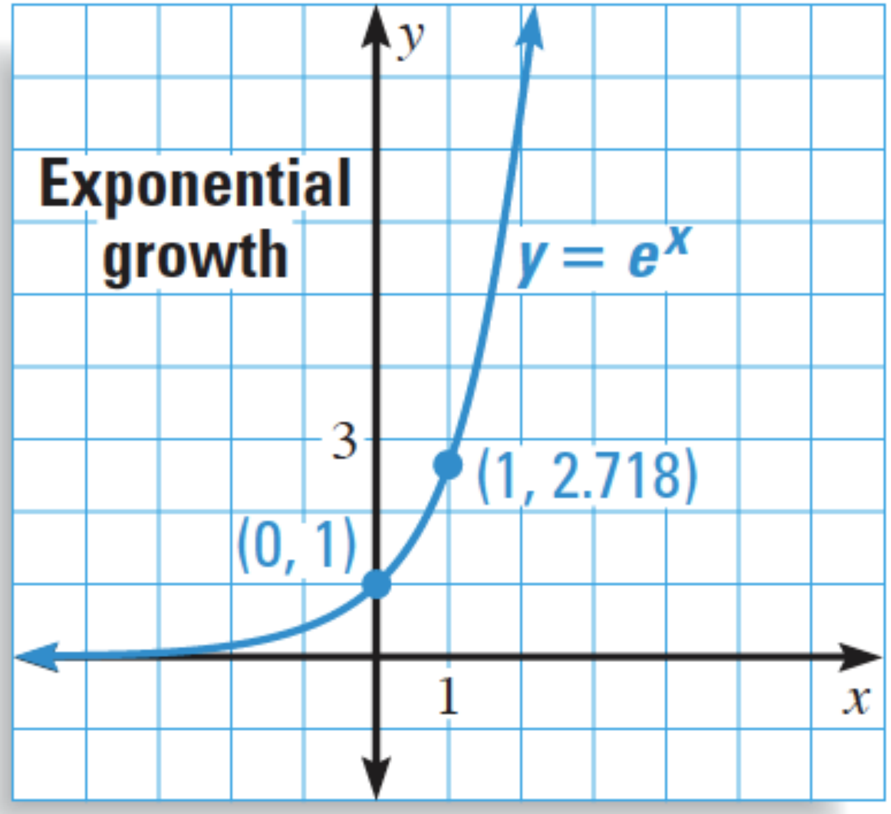
The natural base *e* is irrational. It is defined as follows:

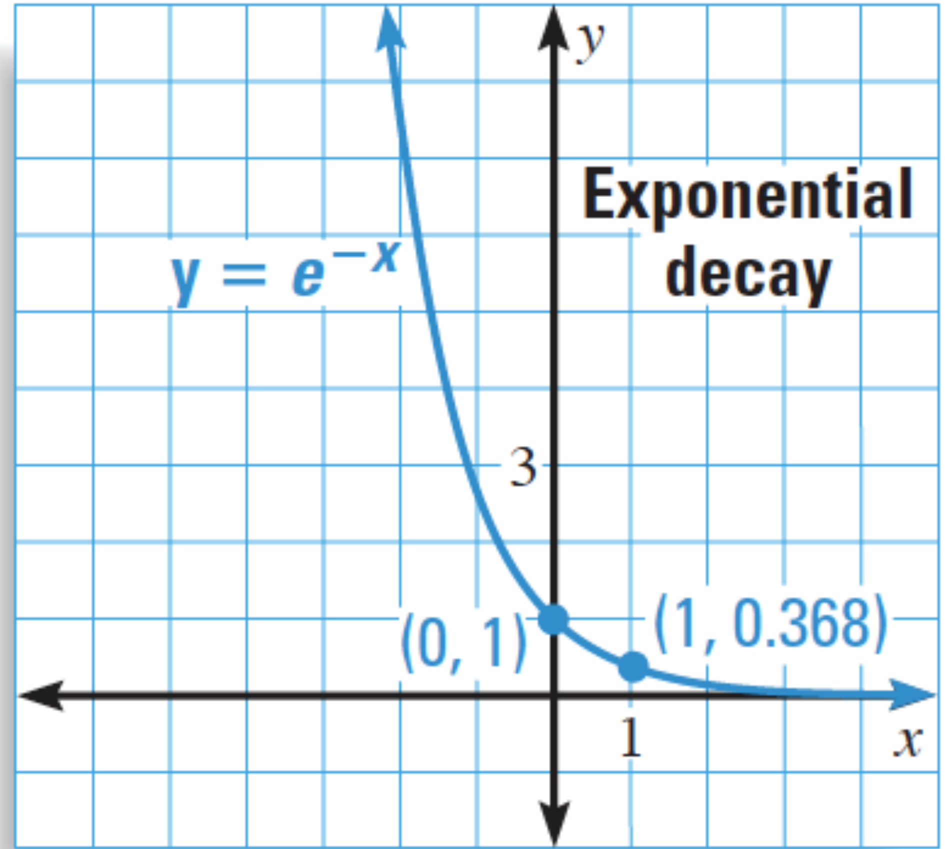
$$\text{As } n \text{ approaches } +\infty, \left(1 + \frac{1}{n}\right)^n \text{ approaches } e \approx 2.718281828459.$$

Graphing $f(x) = ae^{rx}$

- Look at "r" to determine if the function is growth or decay.
- Use the same table of values
- $e = 2.718$ and $\frac{1}{e} = 0.368$

x	-1	0	1
y	1/b	1	b



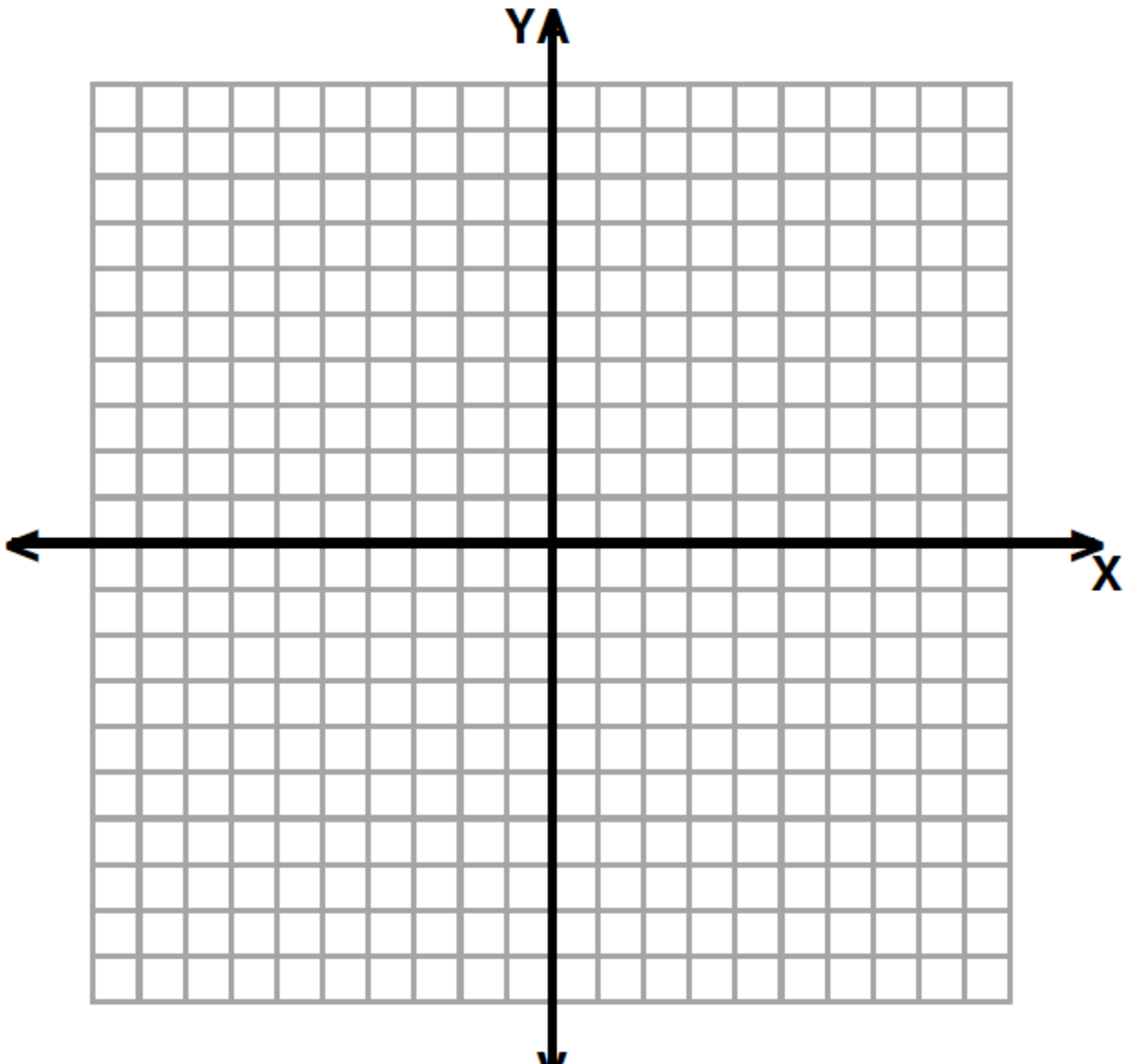


Graphing functions with *e*

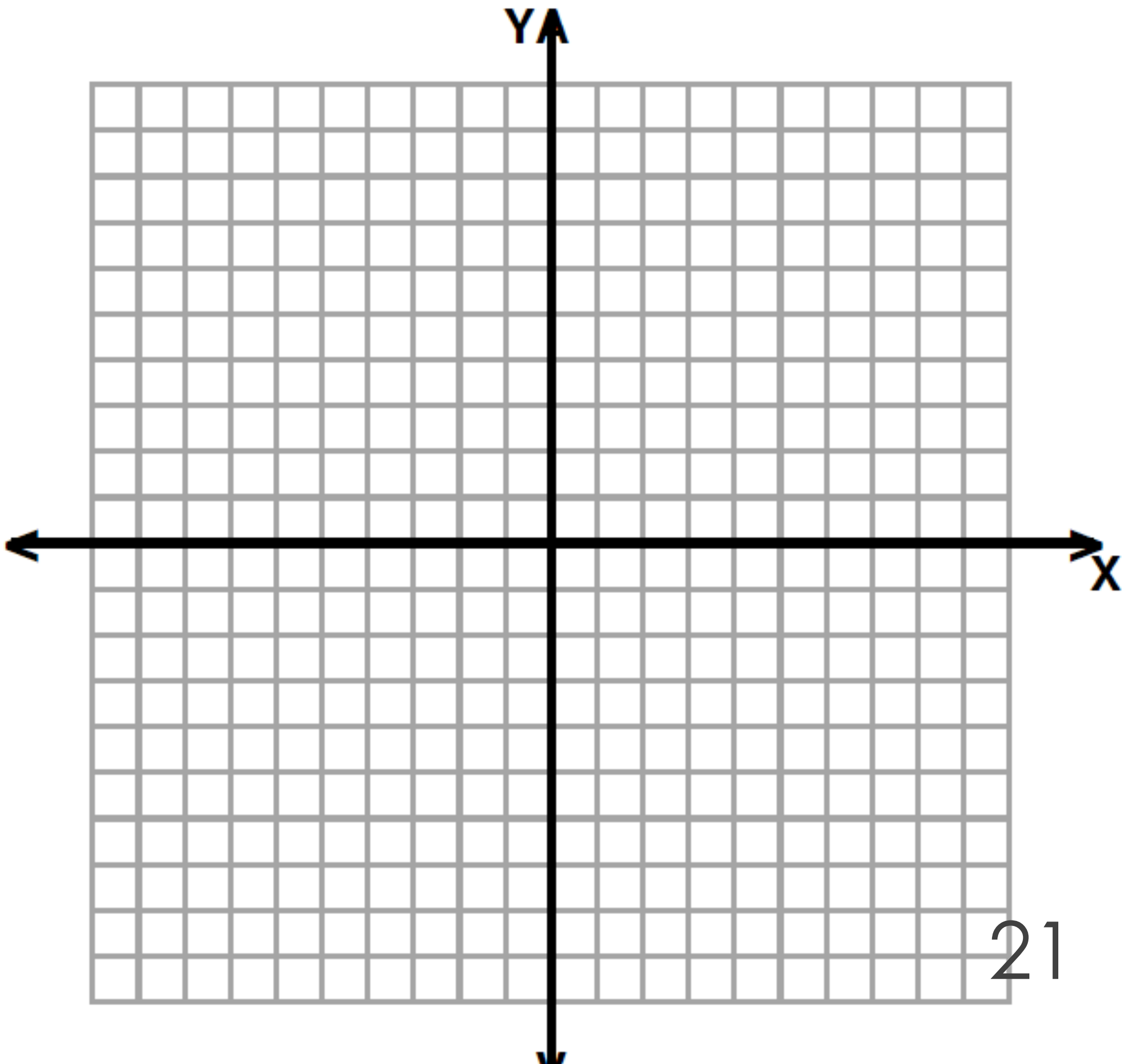
- Look at "a" and the exponent to see if it is a growth or decay function.
  - A and exponent are
- fdsfg

Graph the function. State the domain and range.

a. $y = 2e^{0.75x}$



b. $y = e^{-0.5(x-2)} + 1$



Use of e in real life

- Recall from 8.1, we can calculate compound interest using the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$.
- When we compound interest continuously, the same formula yields $A = Pe^{rt}$.

You deposit $1000 in an account that pays 8% annual interest compounded continuously. What is the balance after 1 year?

CONTINUOUS COMPOUNDING You deposit \$975 in an account that pays 5.5% annual interest compounded continuously. What is the balance after 6 years?

8.4 – LOGARITHMIC FUNCTION

Solve the following equations:

$2^x = 32$

$3^x = 9$

$3^x = \frac{1}{27}$

$5^x = 125$

Definition: Logarithms

Logarithms are the inverse of exponentials. They answer the question "c to what power gives x?"

Logarithmic form	**Exponential Form**
$log_c a = x$	$c^x = a$
$log_5 625 = x$	
	$35^x = 8$
$log_7 49 = x$	
	$5^x = 100$

Special log values

SPECIAL LOGARITHM VALUES

Let b be a positive real number such that $b \neq 1$.

LOGARITHM OF 1	$\log_b 1 = 0$ because $b^0 = 1$.
LOGARITHM OF BASE *b*	$\log_b b = 1$ because $b^1 = b$.

Evaluate the expression.

a. $\log_3 81$

b. $\log_5 0.04$

c. $\log_{1/2} 8$

d. $\log_9 3$

Common and natural logs

COMMON LOGARITHM

$$\log_{10} x = \log x$$

NATURAL LOGARITHM

$$\log_e x = \ln x$$

Exponential and log as inverse

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

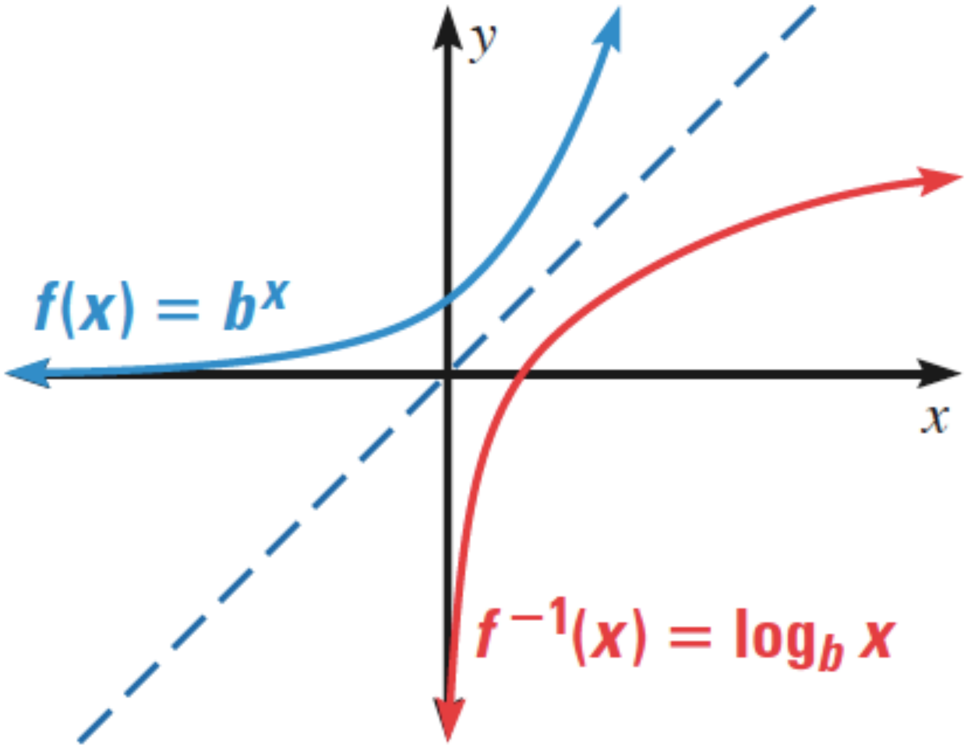
Simplify the expression.

a. $10^{\log 2}$

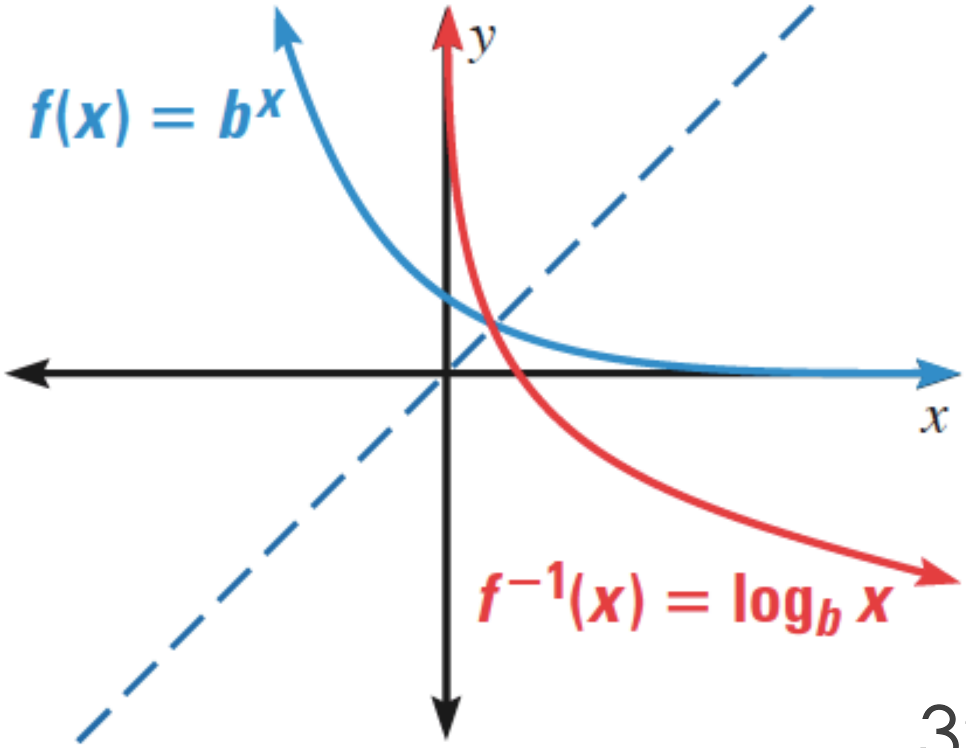
b. $\log_3 9^x$

Exponential and log as inverse

Graphs of f and f^{-1} for $b > 1$



Graphs of f and f^{-1} for $0 < b < 1$



Basic logarithmic function

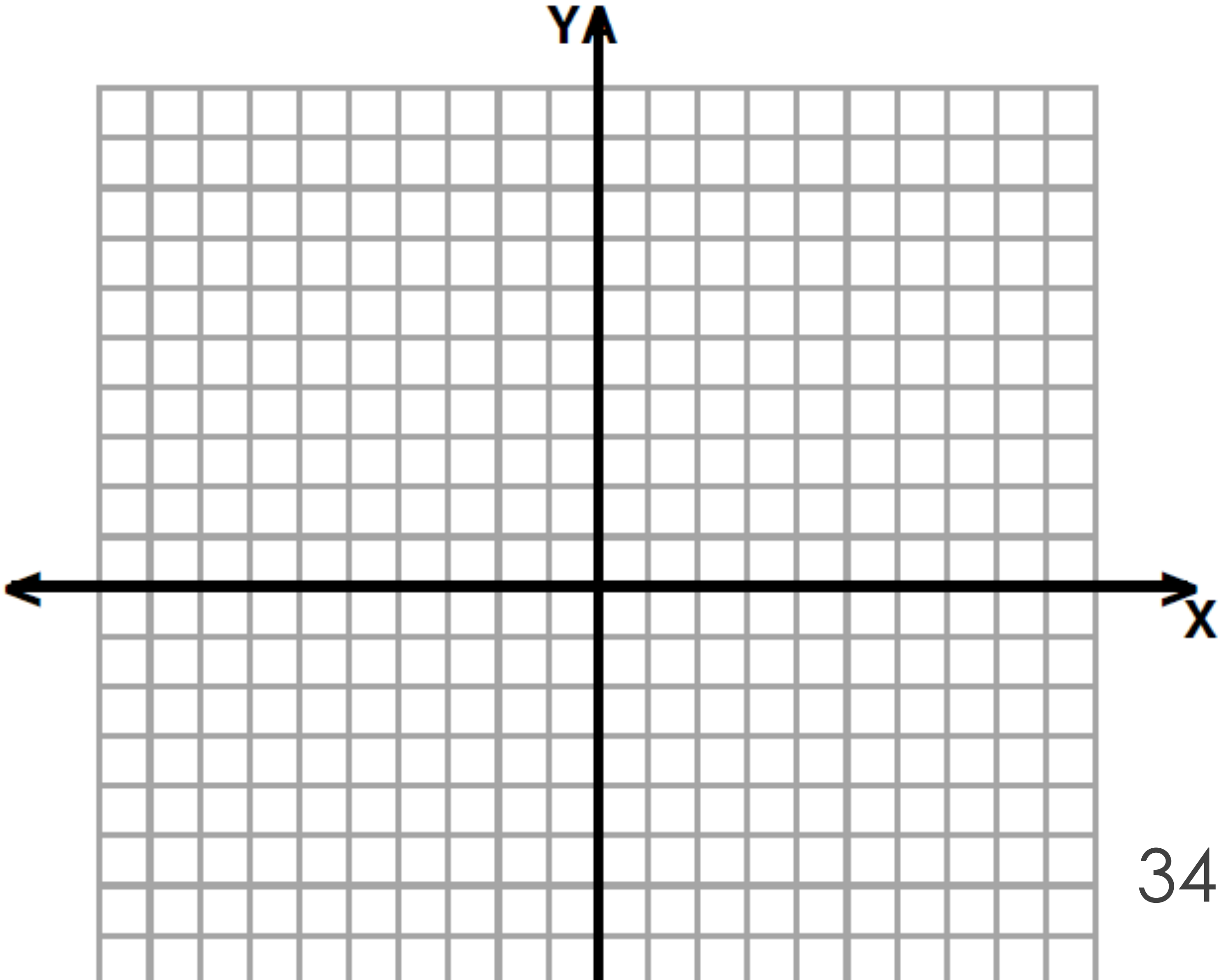
$$f(x) = log_b x$$

◦ It has a vertical **asymptote** because zero

Effect of “b”

Graph $f(x) = log_2 x$

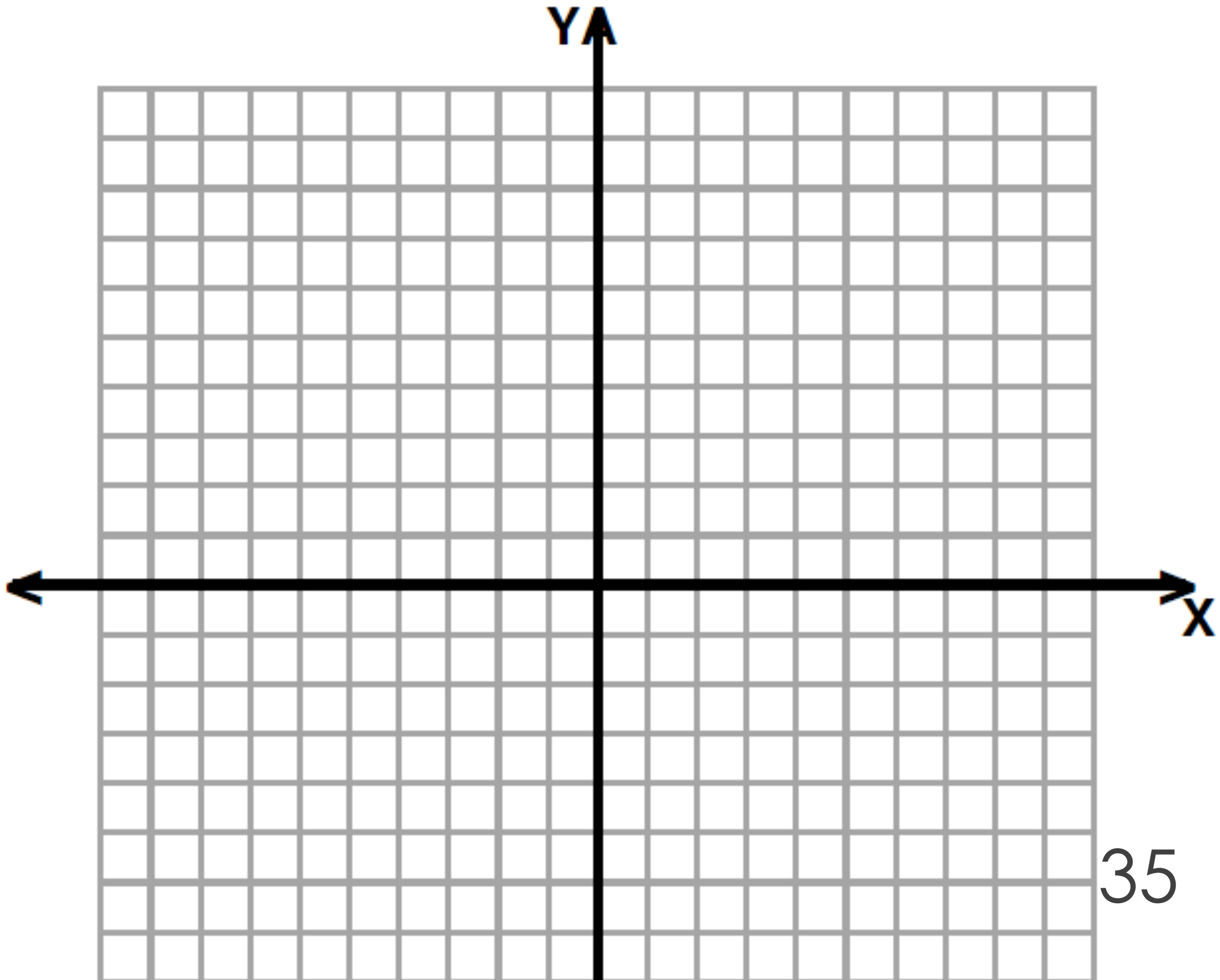
x	$f(x) = 2^x$
1/2	
1	
2	



Effect of "b"

Graph $f(x) = log_{\frac{1}{2}}x$

x	$f(x) = 2^x$
1/2	
1	
2	



Steps for graphing $f(x) = alog_b(x-h) + k$

1) Create a table of values for the basic functions, using -1,0, and 1 for x.

x	1/b	1	b
y	-1	0	1

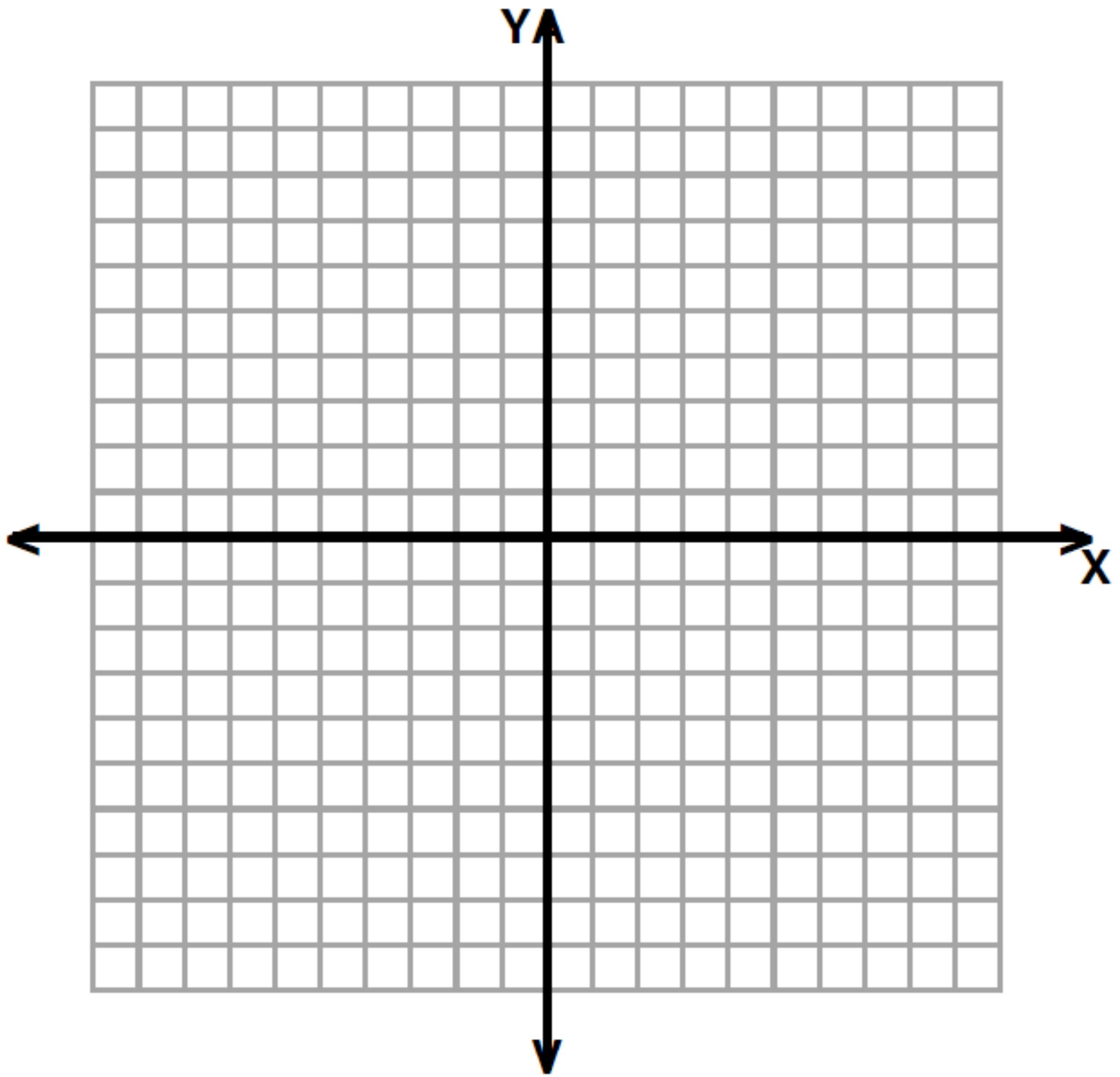
2) Apply the transformations using a, h, and k to get the actual points and create a new table of values.

x→ x+h

y→ ay+k

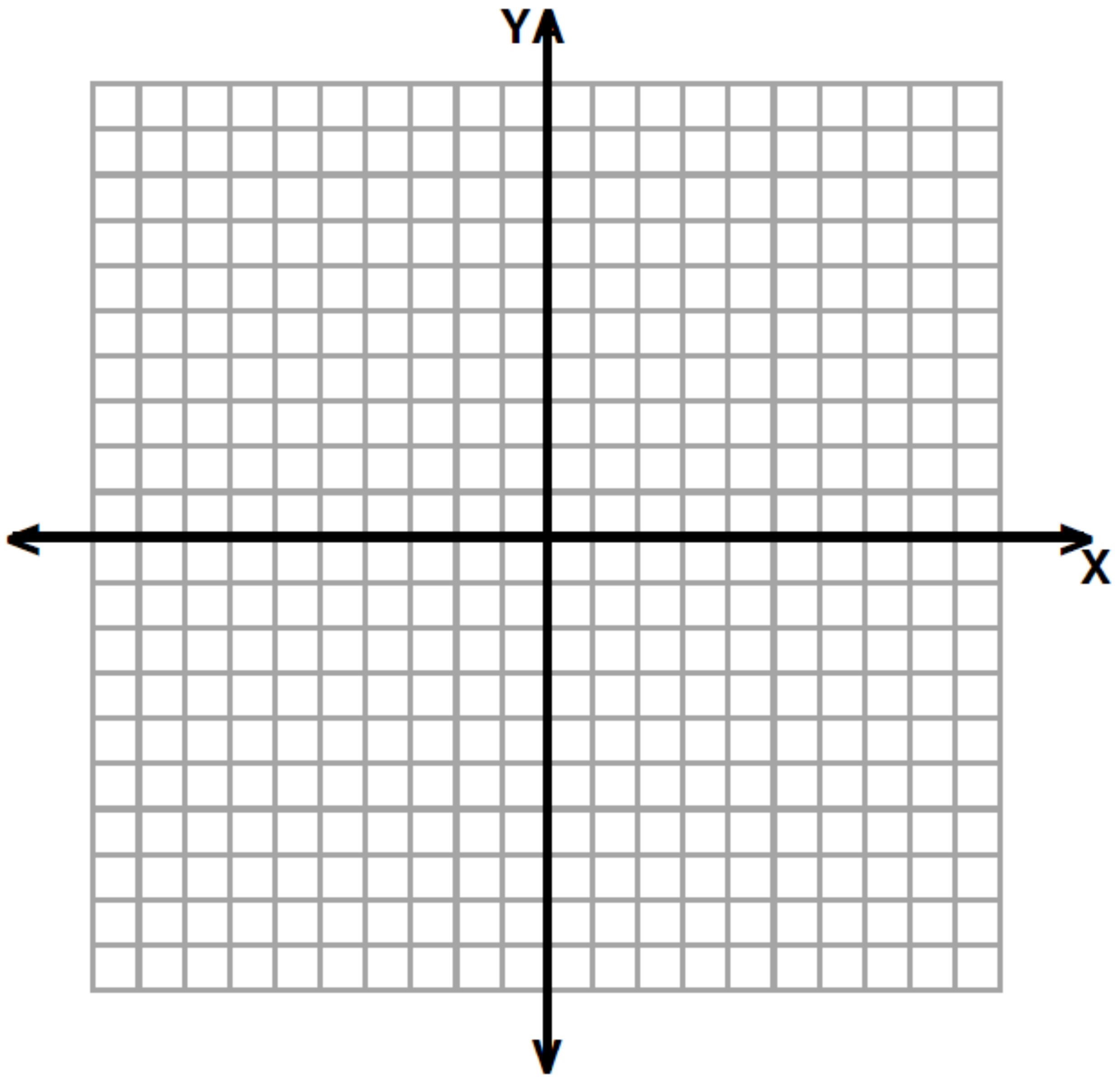
Parameters "h" and "k"

$$y = \log_{1/3} x - 1$$



Parameters “h” and “k

$y = \log_5 (x + 2)$



8.5 – PROPERTIES OF LOGARITHMS

Properties of logarithms

PROPERTIES OF LOGARITHMS

Let b, u, and v be positive numbers such that $b \neq 1$.

PRODUCT PROPERTY	$\log_b uv = \log_b u + \log_b v$
QUOTIENT PROPERTY	$\log_b \frac{u}{v} = \log_b u - \log_b v$
POWER PROPERTY	$\log_b u^n = n \log_b u$

Use $\log_5 3 \approx 0.683$ and $\log_5 7 \approx 1.209$ to approximate the following.

a. $\log_5 \frac{3}{7}$ **b.** $\log_5 21$ **c.** $\log_5 49$

Expanding logs

Expand $\log_2 \frac{7x^3}{y}$. Assume x and y are positive.

$\ln 3xy^3$

$\log_8 64x^2$

Condensing logs

Condense $\log 6 + 2\log 2 - \log 3$.

$7\log_4 2 + 5\log_4 x + 3\log_4 y$

$3(\ln 3 - \ln x) + (\ln x - \ln 9)$

Change of base

CHANGE-OF-BASE FORMULA

Let u, b, and c be positive numbers with $b \neq 1$ and $c \neq 1$. Then:

$$\log_c u = \frac{\log_b u}{\log_b c}$$

In particular, $\log_c u = \dfrac{\log u}{\log c}$ and $\log_c u = \dfrac{\ln u}{\ln c}$.

This property is useful to plug in logarithms in your calculator, which only has the natural and common logarithms.

Change of base proof (not on test)

Using the change of base formula

Evaluate the expression $\log_3 7$ using common and natural logarithms.

$\log_7 12$

8.6 – SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

Types of equations

Exponential	Logarithmic
→Same base, or can get to the same base. →Different base.	→ Both sides have log with same base. → Only one side has a log.

Equivalent values

For each mathematical sentence, determine the value of x.

Sentence	Value of x
$2 \times x = 2 \times 5$	$x =$ ____
$x \div 4 = 8 \div 4$	$x =$ ____
$x + 10 = 6 + 10$	$x =$ ____

Equal powers property

For $b > 0$ and $b \neq 1$,
if $b^x = b^y$, then $x = y$

Ex: If $3^x = 3^5$ then $x = 5$

Equal powers property

- For each mathematical sentence, determine the value of x.

Sentence	Value of x
$5^2 = 5^x$	$x =$ _____
$2^5 = 2^x$	$x =$ _____
$7^3 = 7^x$	$x =$ _____

Exponential equations: Same base

Examples:

$3^{2x+4} = 3^{10}$	$4^{3x-6} = 2^{18}$

Recap: Solving exponential equations

1. Rewrite both sides of the equation with a common base.
2. Set the exponents on each side equal to each other.
3. Solve the equation.
4. Check your answer.

Equal logarithms property

For positive numbers b, x and y where $b \neq 1$,

$\log_b x = \log_b y$ if and only if $x = y$

Ex: $\log_3 x = \log_3 5$ if and only if $x = 5$

Equal Logarithms property

For each mathematical sentence, determine the value of x.

Sentence	Value of x
$\log_5(2) = \log_5(x)$	$x =$ _____
$\log_2(5) = \log_2(x)$	$x =$ _____
$\log_7(3) = \log_7(x)$	$x =$ _____

Log equations: same base

Examples:

$\log_3(2x+4) = \log_3(3x+3)$	$\log_5(3x-4) - \log_5(x) = \log_5(2)$

Recap: Solving Logarithmic equations

1. Rewrite both sides of the equation as one log with common base.
2. Set the insides of the log on each side equal to each other.
3. Solve the equation.
4. Check your answer - remove extraneous solutions.

Take the common log of each side

When you are unable to write both sides of the equation with a common base, you can take the common logarithm of each side.

Use the power rule to bring the exponent down.

$$\log_b(x^m) = m\log_b(x)$$

Exponential equations: different base

Examples:

$2^x = 20$	$5^{2x-3} = 16$

Recap: Solving exponential equations with different bases.

1. Take the common log of both sides.
2. Use the power rule to bring the exponent down.
3. Solve the equation.
4. Check your answer.

Exponentiate each side

When only one side is a logarithmic expression, exponentiate both sides.

- Use the fact that log and exponents are inverses to cancel the log out.

$$b^{\log_b x} = x$$

Log equations: log on one side only.

Examples:

$\log_2(4x + 3) = 3$	$2\log_3(2x + 1) = 4$

Recap: Solving Logarithmic equations

1. Exponentiate both sides.
2. Solve the equation.
3. Check your answer - remove extraneous solutions.