CHAPTER 8

Exponential and Logarithmic functions

8.1/8.2 - EXPONENTIAL GROWTH AND DECAY

Basic exponential function

$$f(x) = b^x$$

base

$b > 0$

$y = 0$

- It has a horizontal **asymptote** because it will never reach zero, no matter what x-value you put in the function.

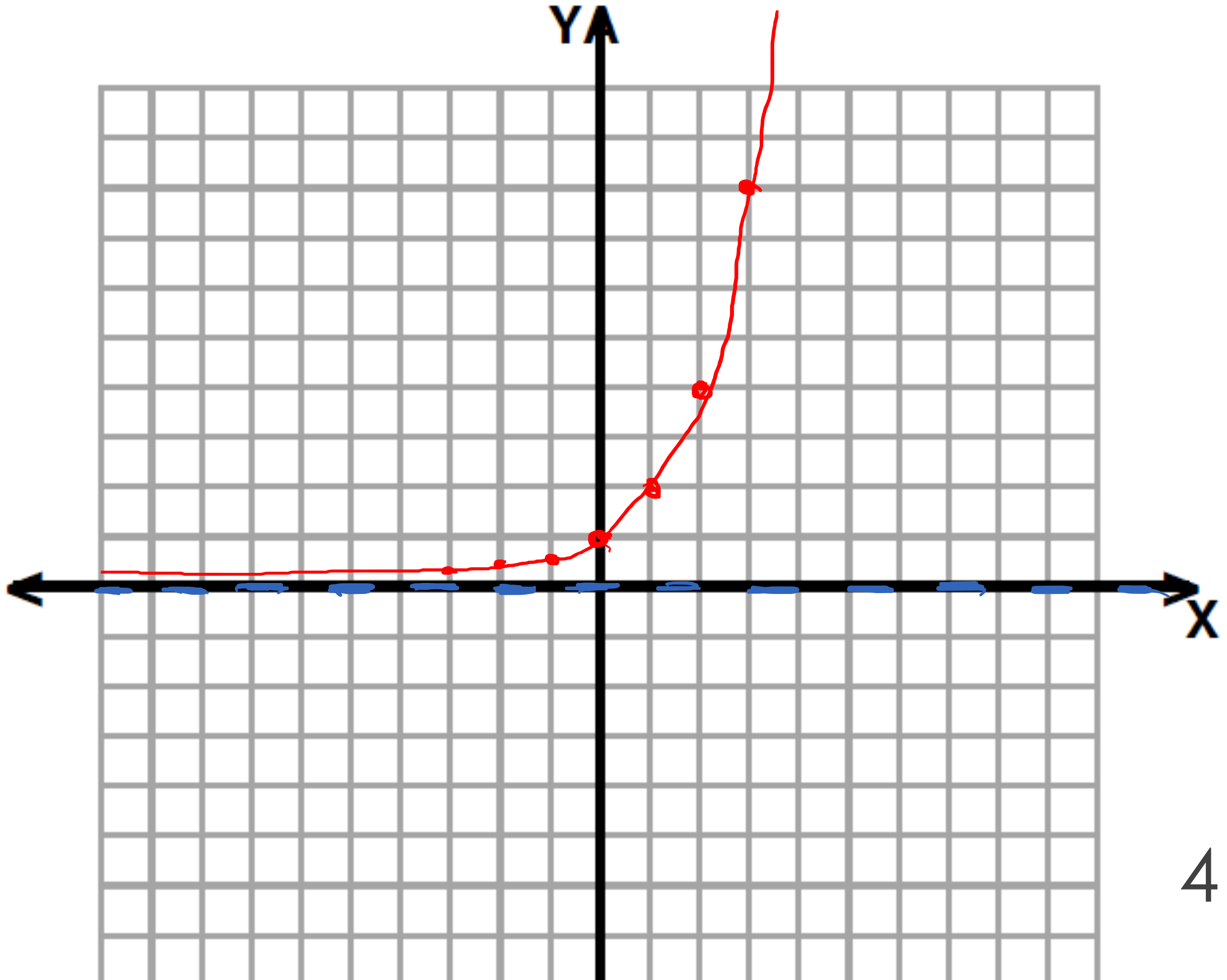
asymptote = a line that a function gets infinitely close to but never touches.

Growth vs. Decay

Growth Function

Graph $f(x) = 2^x$

x	$f(x) = 2^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

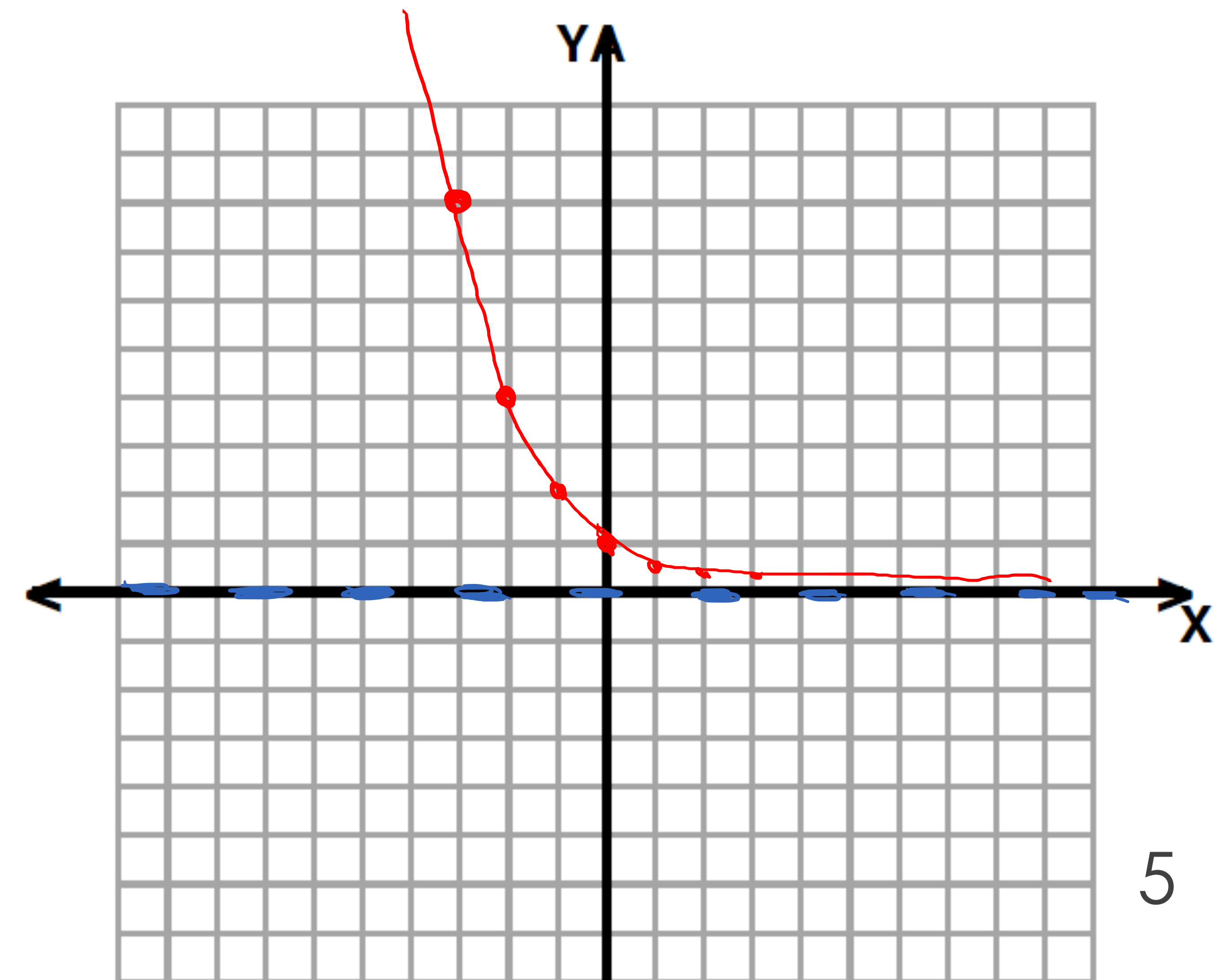


Growth vs. Decay

Decay

Graph $f(x) = \left(\frac{1}{2}\right)^x$

x	$f(x) = 2^x$
-3	$\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$
0	$\left(\frac{1}{2}\right)^0 = 1$
1	$\left(\frac{1}{2}\right)^1 = 1/2$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$



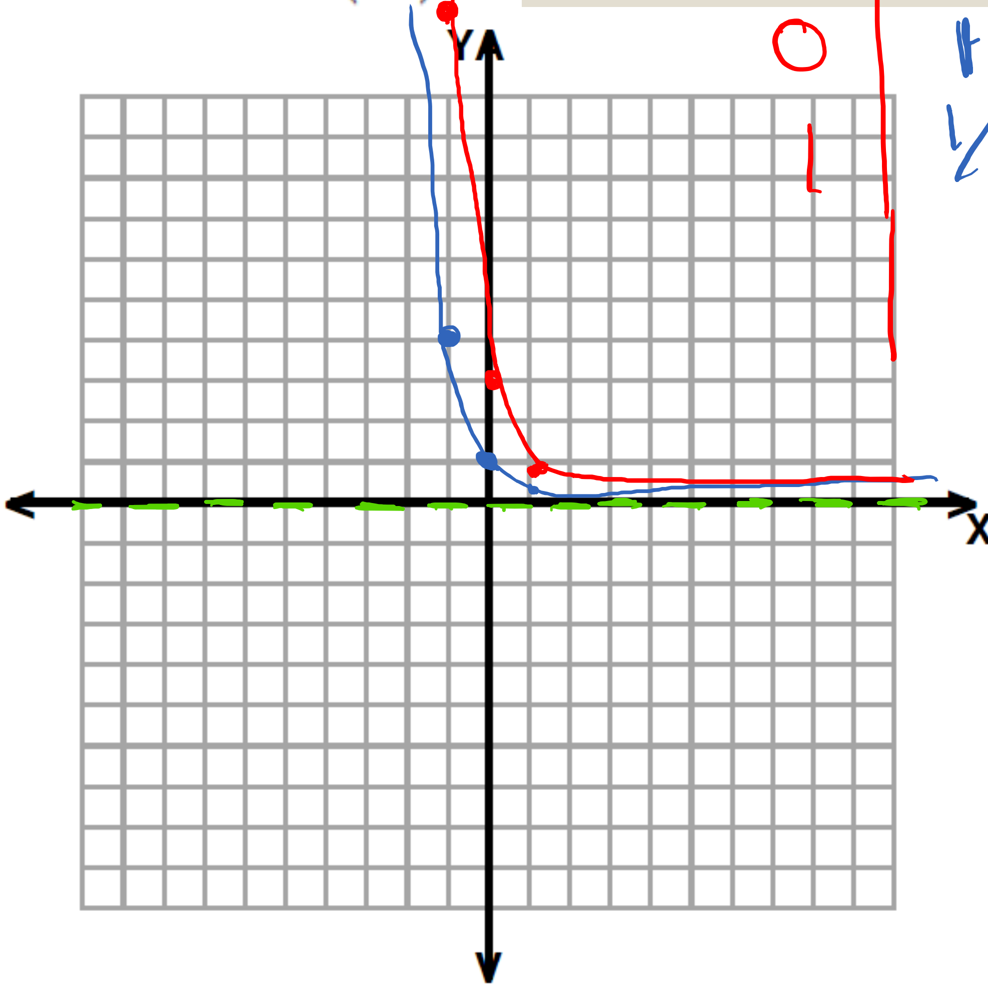
Growth vs. Decay

Growth	Decay
• Base $b > 1$ • Approaches the asymptote on the left.	• Base $0 < b < 1$ • Approaches the asymptote on the right.

Parameter "a" $f(x) = a \cdot b^x$

a. $y = 3\left(\frac{1}{4}\right)^x$

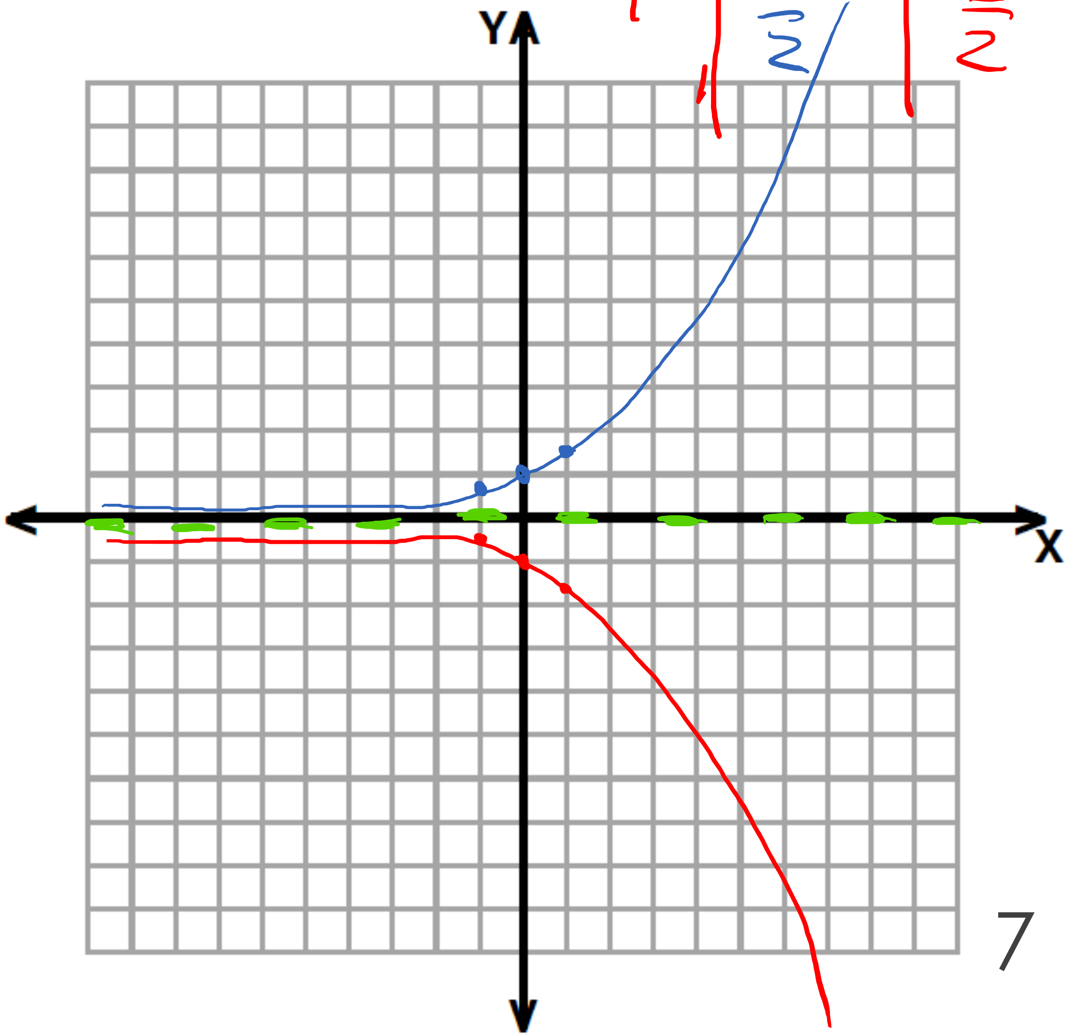
x	$\left(\frac{1}{4}\right)^x$	$3\cdot\left(\frac{1}{4}\right)^x$
-1	4	12
0	1	3
1	$1/4$	$\frac{3}{4}$ 0.75



x	y
-1	$1/6$
0	1
1	6

b. $y = -\left(\frac{3}{2}\right)^x$

x	$\left(\frac{3}{2}\right)^x$	$-\left(\frac{3}{2}\right)^x$
-1	$\frac{2}{3}$	$-\frac{2}{3}$
0	1	-1
1	$\frac{3}{2}$	$-\frac{3}{2}$



Steps for graphing $f(x) = a \cdot b^{(\underline{x-h})} + k$

1) Create a table of values for the basic functions, using -1,0, and 1 for x.

x	-1	0	1
y	1/b	1	b

$\left.\begin{matrix}(-1, \frac{1}{b}) \\ (0, 1) \\ (1, b)\end{matrix}\right\}$ for base functions

2) Apply the transformations using a, h, and k to get the actual points and create a new table of values.

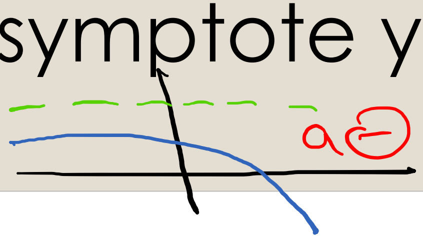
x→ x+h

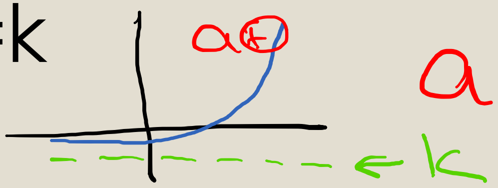
y→ ay+k

Dom: $\mathbb{R}$ all real numbers

Range: $\underbrace{]-\infty, k[}_{\text{a positive}}$ or $\underbrace{]k, +\infty[}_{\text{a negative}}$

3) Draw the asymptote y=k





Parameters "h" and "k" $f(x) = a \cdot b^{(x-h)} + k$

$a=3$ $h=1$ $k=-4$

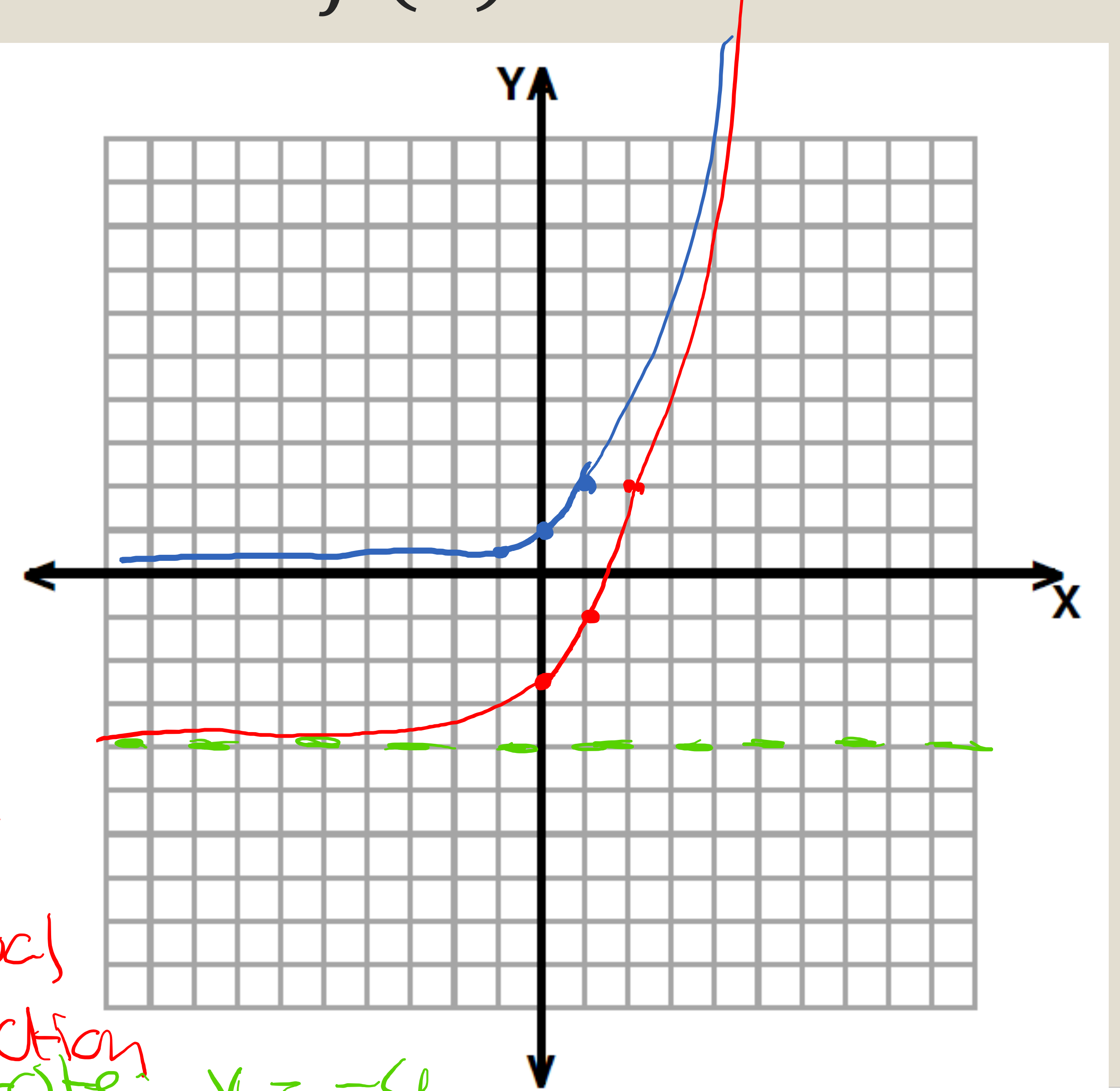
$$y = 3 \cdot 2^{x-1} - 4$$

x	$x+1$	2^x	$3 \cdot 2^{x-1} - 4$
-1	$-1+1$ 0	$1/2$	$3(\frac{1}{2})-4$ -2.5
0	$0+1$ 1	1	$3(1)-4$ -1
1	$1+1$ 2	2	$3(2)-4$ 2

$(-1, \frac{1}{2})$
$(0, 1)$
$(1, 2)$
base

$(0, -2.5)$
$(1, -1)$
$(2, 2)$
actual function

asymptote: $y=-4$



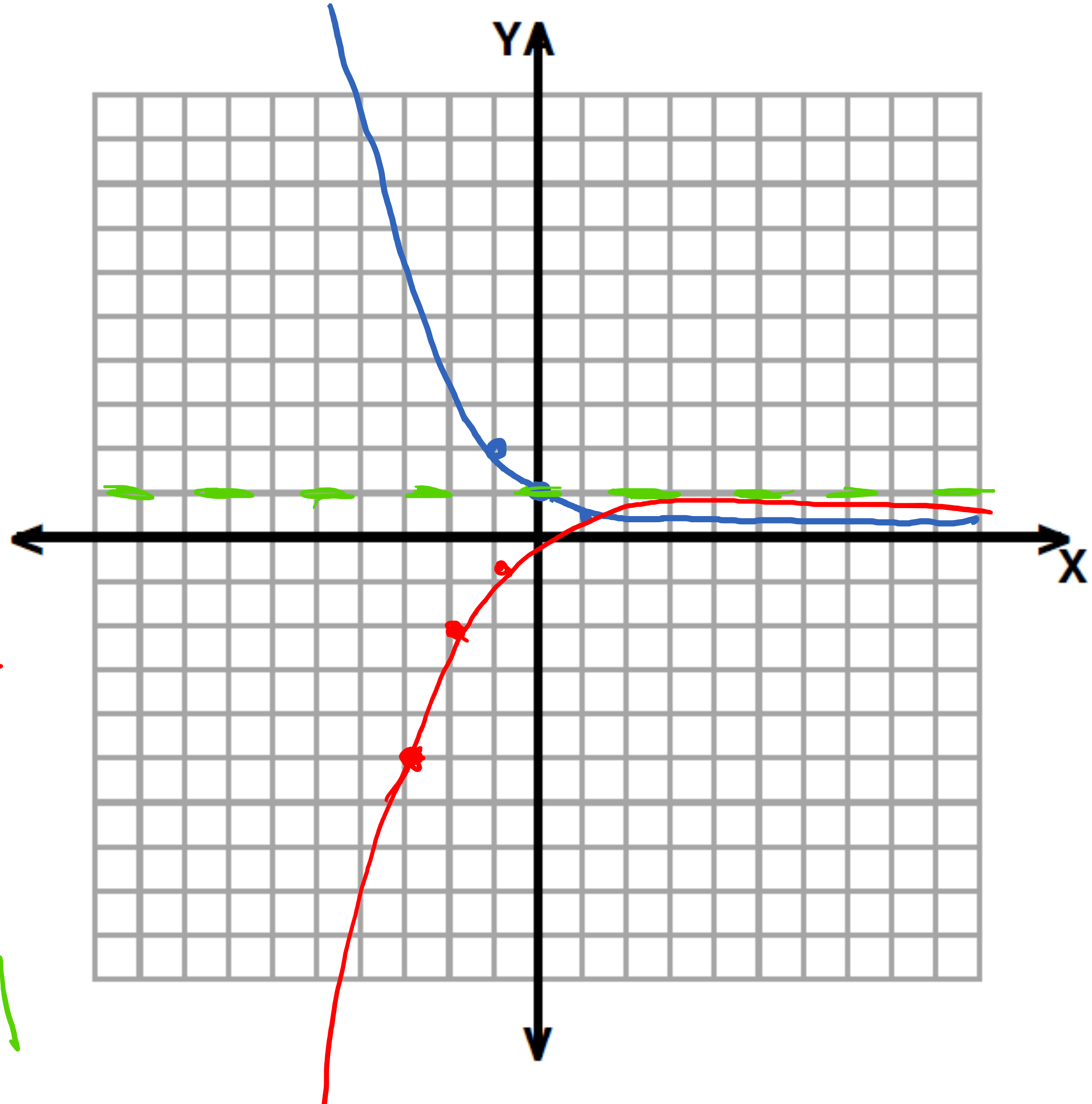
Parameters "h" and "k" $f(x) = a \cdot b^{(x-h)} + k$

$a = -3$ $h = -2$ $k = 1$

$y = -3\left(\frac{1}{2}\right)^{x+2} + 1$

x	$\left(\frac{1}{2}\right)^x$	$x-2$	$-3\left(\frac{1}{2}\right)^{x+2} + 1$
-1	2	-3	$-3(2)+1$ -5
0	1	-2	$-3(1)+1$ -2
1	$\frac{1}{2}$	-1	$-3(\frac{1}{2})+1$ -0.5

asymptote $y=1$



Exponentials to represent growth or decay.

When a real-life quantity increases by a fixed percent each year, the quantity can be modeled by:

$$y = a(1 + r)^t$$

a = initial value at time t = 0
t = time gone by
r = rate of increase or decrease.
y = value at time t

For decay: $y = a(1 - r)^t$

In percent:
50% more → 150% of initial
20% off → 80% of initial value

$a = 1.313M$

INTERNET HOSTS In January, 1993, there were about 1,313,000 Internet hosts. During the next five years, the number of hosts increased by about 100% per year.

$r = 1$

▶ Source: Network Wizards

$y = a(1+r)^t$

a) $h = 1.313(1+1)^t$

$h = 1.313 \cdot 2^t$

1996 : $t = 3$ $h = 1.313(2)^3$

$h = 10.504$ M

a. Write a model giving the number h (in millions) of hosts t years after 1993. About how many hosts were there in 1996?

↳ $t = 3$

b. Graph the model.

c. Use the graph to estimate the year when there were 30 million hosts.

b)

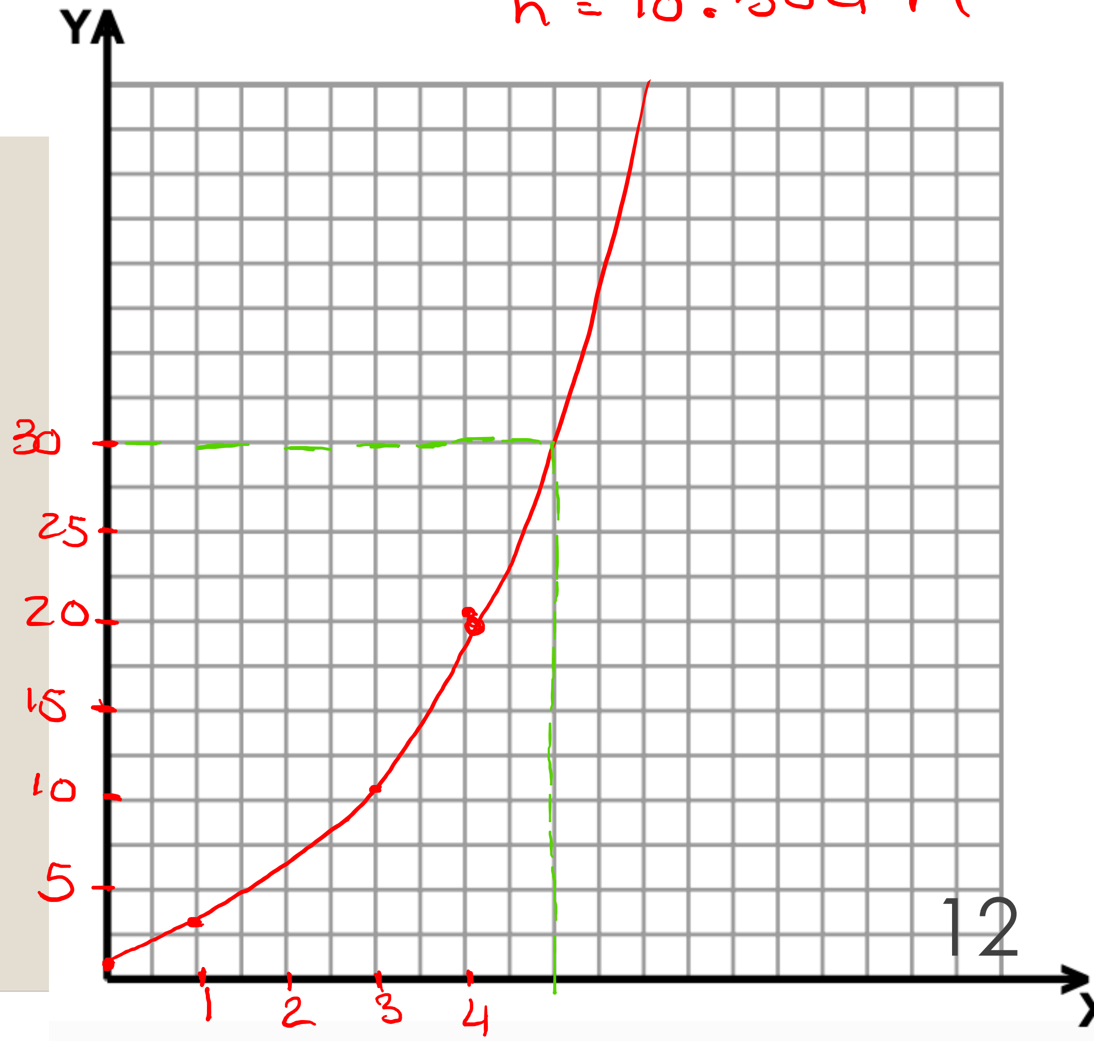
t	h
0	1.313
1	2.626
3	10.504
4	21.008

→ $h = 1.313 \cdot 2^1 = 2.626$

→ $h = 1.313 \cdot 2^4 = 21.008$

c) 4.5 – 5 yrs.

GROWTH



You buy a new car for \$24,000. The value y of the car decreases by 16% each year.

$a = 24\,000$ $\quad r = 0.16$

a. Write an exponential decay model for the value of the car. Use the model to estimate the value after 2 years.

b. Graph the model.

c. Use the graph to estimate when the car will have a value of \$12,000.

DECAY

$y = a(1-r)^t$

a) $y = 24000(1-0.16)^t$

$y = 24000(0.84)^t$

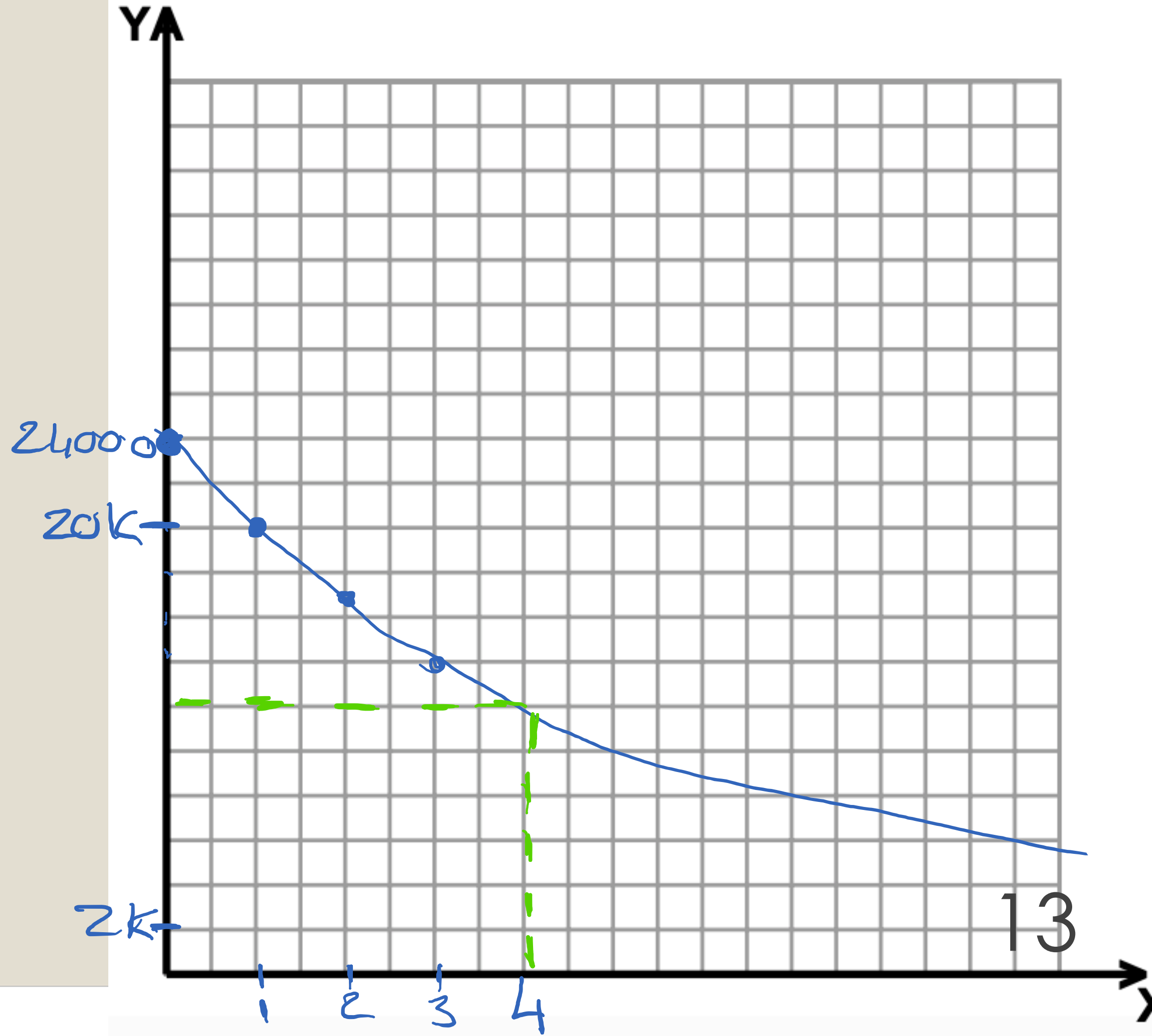
at $t = 2$ yrs

$y = 24000(0.84)^2 = 16934.40$

b)

x	y
0	24000
1	20160
2	16934.40
3	14245

c) about 4 years.



Compound interest

COMPOUND INTEREST

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years can be modeled by this equation:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

n= # of times that interest is compounded per year.

P= initial value

A= amout at a given time

t= time gone by

FINANCE You deposit \$1000 in an account that pays 8% annual interest. Find the balance after 1 year if the interest is compounded with the given frequency.

$A = P\left(1+\frac{r}{n}\right)^{nt}$

$P = 1000$

$r = 0.08$

a. annually

$n = 1$

$A = 1000\left(1+\frac{0.08}{1}\right)^{1 \cdot t}$

$A = 1000(1.08)^{t}$

$t = 1$

$A = 1000(1.08)^{1}$

$A = \$1080$

b. quarterly

$n = 4$

$A = 1000\left(1+\frac{0.08}{4}\right)^{4t}$

$A = 1000(1.02)^{4t}$

$t = 1$

$A = 1082.43$

c. daily

$n = 365$

$A = 1000\left(1+\frac{0.08}{365}\right)^{365t}$

$A = 1000(1.000219178)^{365t}$

$t = 1$

$A = 1083.28$

8.3 – THE NUMBER e

ACTIVITY

Developing Concepts

Investigating the Natural Base e

1 Copy the table and use a calculator to complete the table.

n	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1+\frac{1}{n}\right)^n$	2.594	2.705	2.717	2.718	2.718	2.718

$\left(1+\frac{1}{10^2}\right)^{10^2}$

2 Do the values in the table appear to be approaching a fixed decimal number? If so, what is the number rounded to three decimal places?

THE NATURAL BASE e

The natural base e is irrational. It is defined as follows:

As n approaches $+\infty$, $\boxed{\left(1 + \frac{1}{n}\right)^n}$ approaches $e \approx 2.718281828459$.

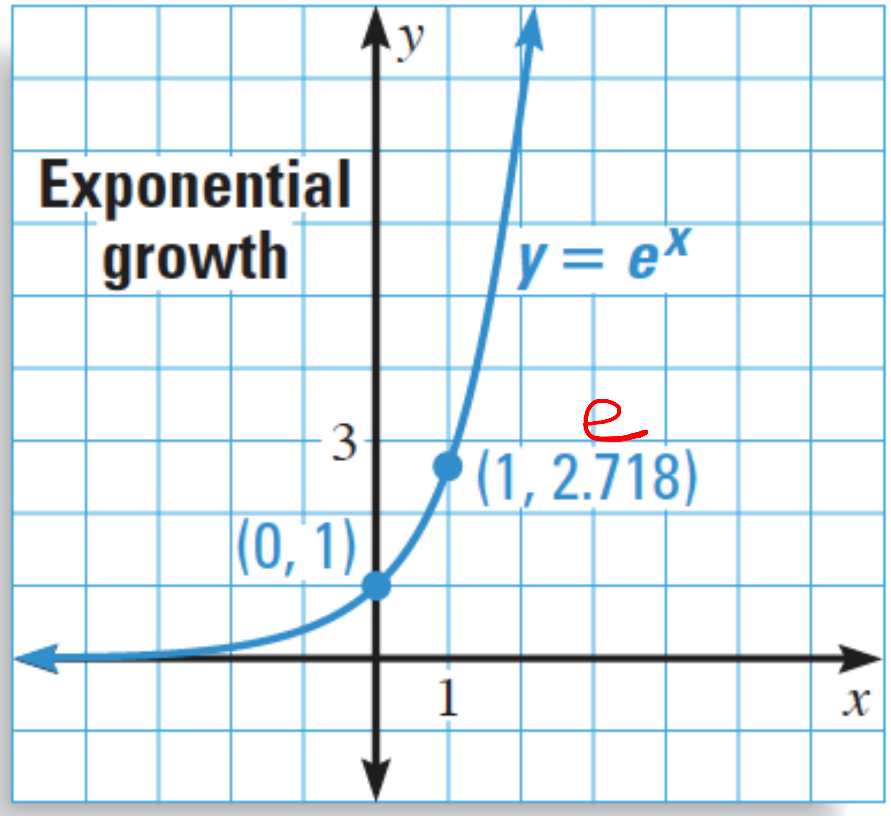
Graphing $f(x) = ae^{rx}$

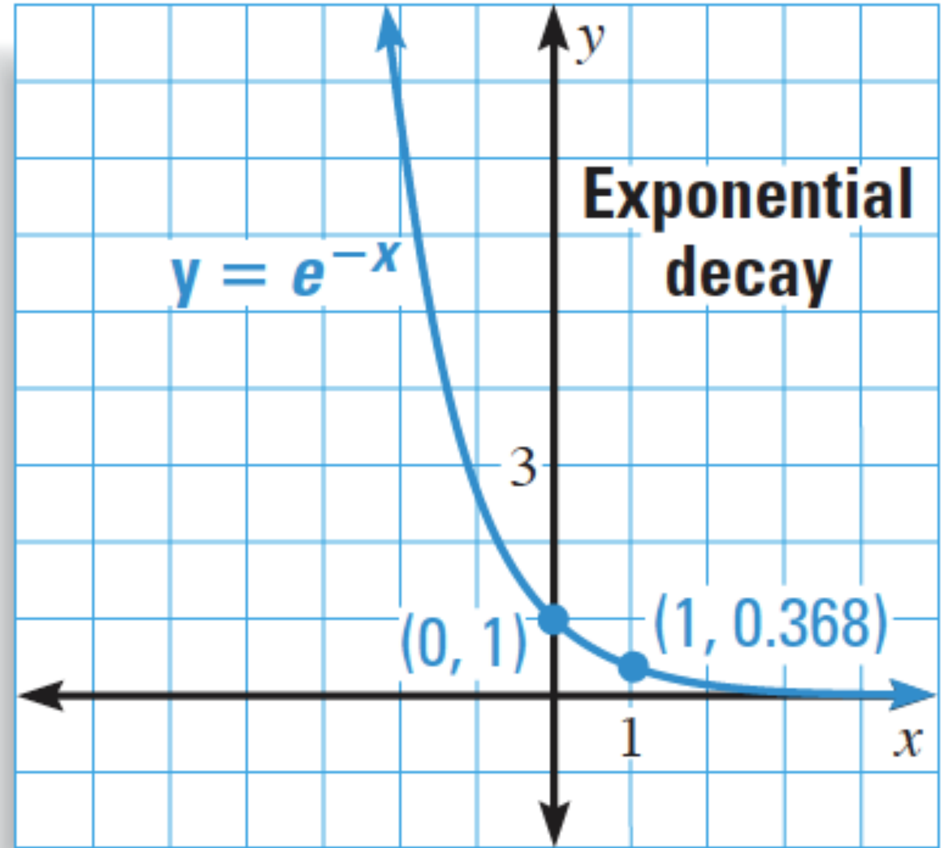
$e^{-x} = \left(\frac{1}{e}\right)^x$

less than 1

- Look at "r" to determine if the function is growth or decay. $r > 0 \to$ growth, $r < 0 \to$ decay
- Use the same table of values
- $e = 2.718$ and $\frac{1}{e} = 0.368$

x	-1	0	1
y	1/b	1	b





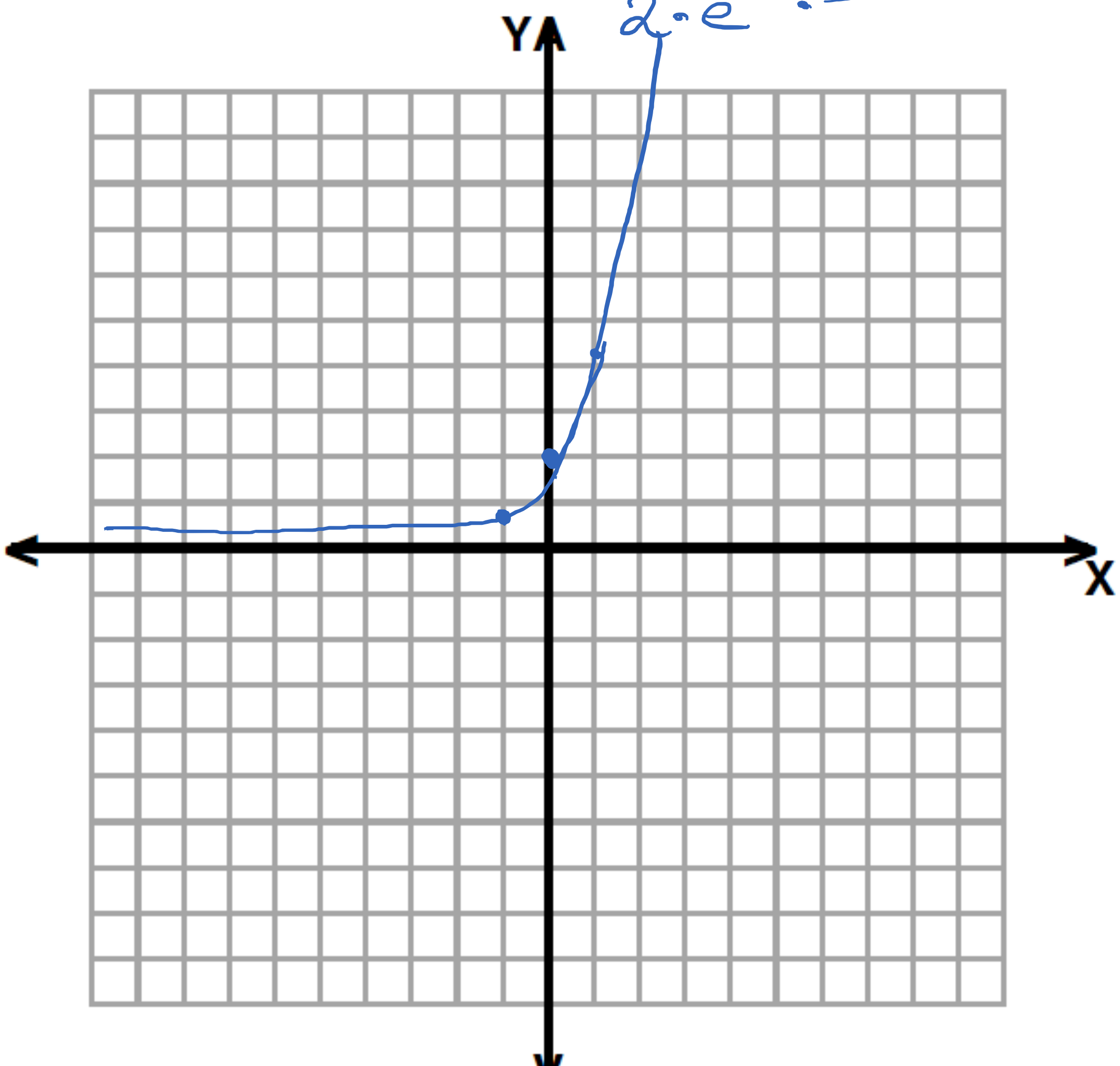
Graph the function. State the domain and range.

growth

a. $y = 2e^{0.75x}$

x	-1	0	1
y	0.94	2	4.23

$2 \cdot e^{-0.75}$



decay

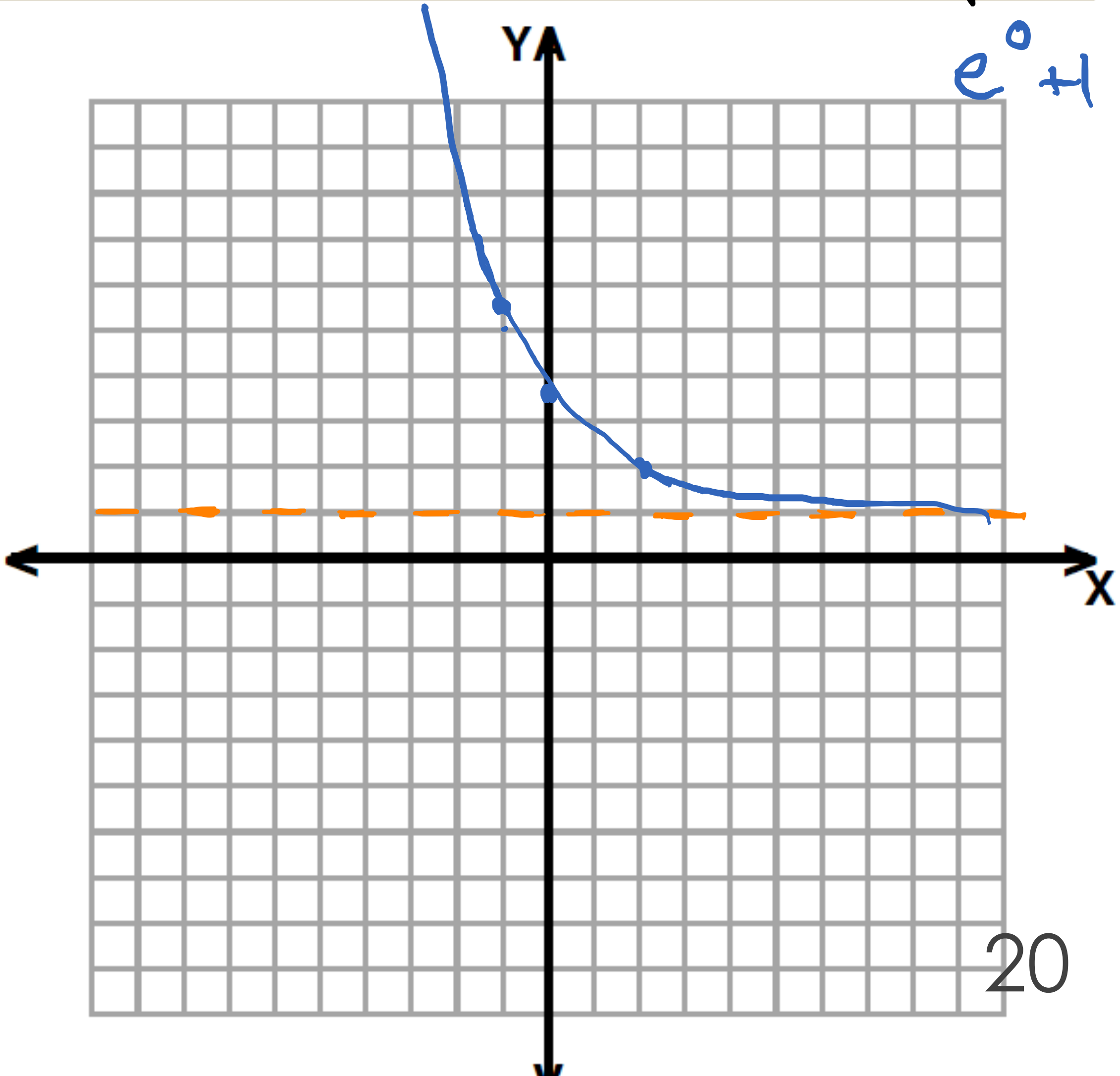
because it yields e^0

b. $y = e^{-0.5(x-2)} + 1$

asymptote

x	-1	0	2
y	5.48	3.72	2

$e^0 + 1$



Use of e in real life

- Recall from 8.1, we can calculate compound interest using the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$.
- When we compound interest continuously, the same formula yields $A = Pe^{rt}$. "Pert"

P = initial value
r = rate
t = # years gone by.

You deposit \$1000 in an account that pays 8% annual interest compounded continuously. What is the balance after 1 year? $A = Pe^{rt}$

$P = 1000$

$r = 8\% = 0.08$

$A = ?$

$t = 1 \text{ yr}$

$A = 1000e^{0.08(1)}$

$A = \$1083.29$

CONTINUOUS COMPOUNDING You deposit \$975 in an account that pays 5.5% annual interest compounded continuously. What is the balance after 6 years?

$A = Pe^{rt}$

$P = \$975$

$r = 5.5\% = 0.055$

$t = 6$ yrs

$A = 975e^{0.055(6)}$

$A = \$1356.19$

8.4 – LOGARITHMIC FUNCTION

Solve the following equations:

$2^x = 32$

$2^x = 2^5$

$x = 5$

$3^x = 9$

$3^x = 3^2$

$x = 2$

$3^x = \frac{1}{27}$

$3^x = (27)^{-1}$

$3^x = (3^3)^{-1}$

$3^x = 3^{-3} \rightarrow x = -3$

$5^x = 125$

$5^x = 5^3$

$x = 3$

Definition: Logarithms

Logarithms are the inverse of exponentials. They answer the question "b to what power gives a?"

$\log_b a = x$

$b^x = a$

Logarithmic form	Exponential Form
$log_b a = x$	$b^x = a$
$log_5 625 = x$	$5^x = 625$
$\log_{35} 8 = x$	$35^x = 8$
$log_7 49 = x$	$7^x = 49$
$\log_5 100 = x$	$5^x = 100$

Special log values

SPECIAL LOGARITHM VALUES

Let b be a positive real number such that $b \neq 1$.

LOGARITHM OF 1 $\log_b 1 = 0$ because $b^0 = 1$.

LOGARITHM OF BASE *b* $\log_b b = 1$ because $b^1 = b$.

Evaluate the expression.

To what power do I raise the base to obtain what is inside the logarithm?

a. $\log_3 81 = x$

$3^x = 81$

$3^4 = 81$

$x = 4$

b. $\log_5 0.04 = x$

$5^x = 0.04$

$\log_5 \frac{1}{25}$

$\log_5 (25)^{-1}$

$\log_5 5^{-2} \to \log_5 0.04 = -2$

c. $\log_{1/2} 8$

$8 = 2^3$

$8 = \left(\frac{1}{2}\right)^{-3}$

$\log_{1/2} \left(\frac{1}{2}\right)^{-3}$

$\log_{\frac{1}{2}} 8 = -3$

d. $\log_9 3$

$\sqrt{a} = a^{1/2}$

$\sqrt{9} = 3$

$\log_9 \sqrt{9}$

$\log_9 9^{1/2}$

$\log_9 3 = \frac{1}{2}$

$\log_b x$ ← b is the base

Common and natural logs

COMMON LOGARITHM

$$\log_{10} x = \log x$$

← 10 does not need to be written.

NATURAL LOGARITHM

$$\log_e x = \ln x$$

$\sqrt{x} = \sqrt[2]{x}$

$$\frac{2x}{2} = \frac{4}{2} \quad x = 2$$

Exponential and log as inverse

$f(x) = b^x$ $\qquad$ $g(x) = \log_b x$

$g(f(x)) = \log_b b^x = x$ $\qquad$ $f(g(x)) = b^{\log_b x} = x$

Simplify the expression.

a. $\cancel{10^{\log_{10} 2}}$ $\qquad$ **b.** $\log_3 9^x$

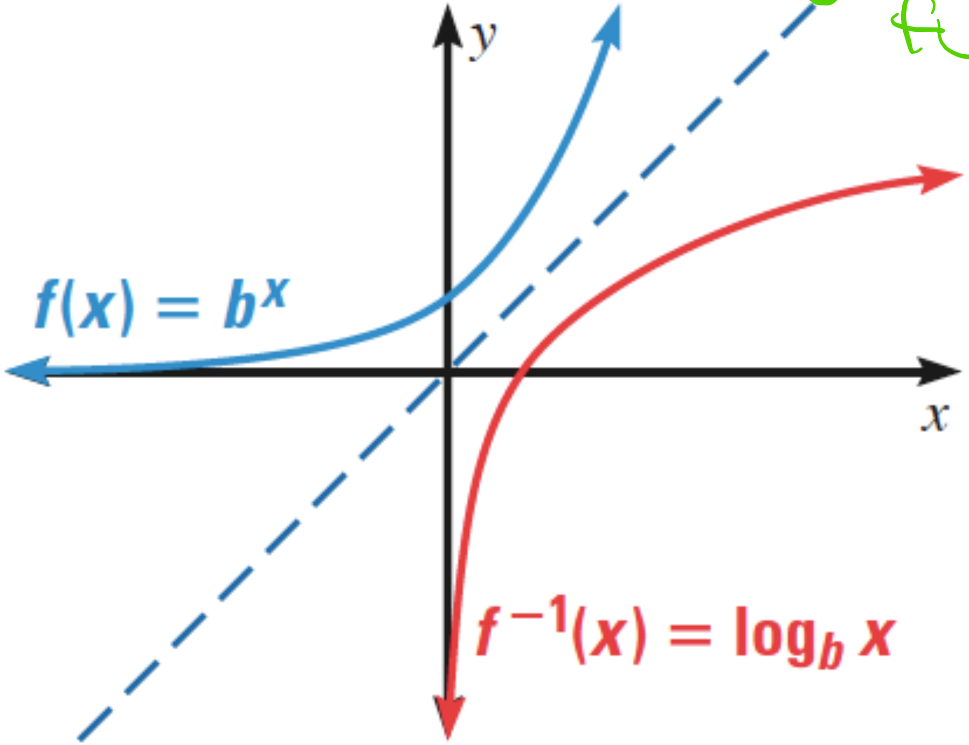
2 $\qquad$ $\log_3 (3^2)^x$

$\cancel{\log_3} \cancel{3}^{2x} = 2x$

Exponential and log as inverse

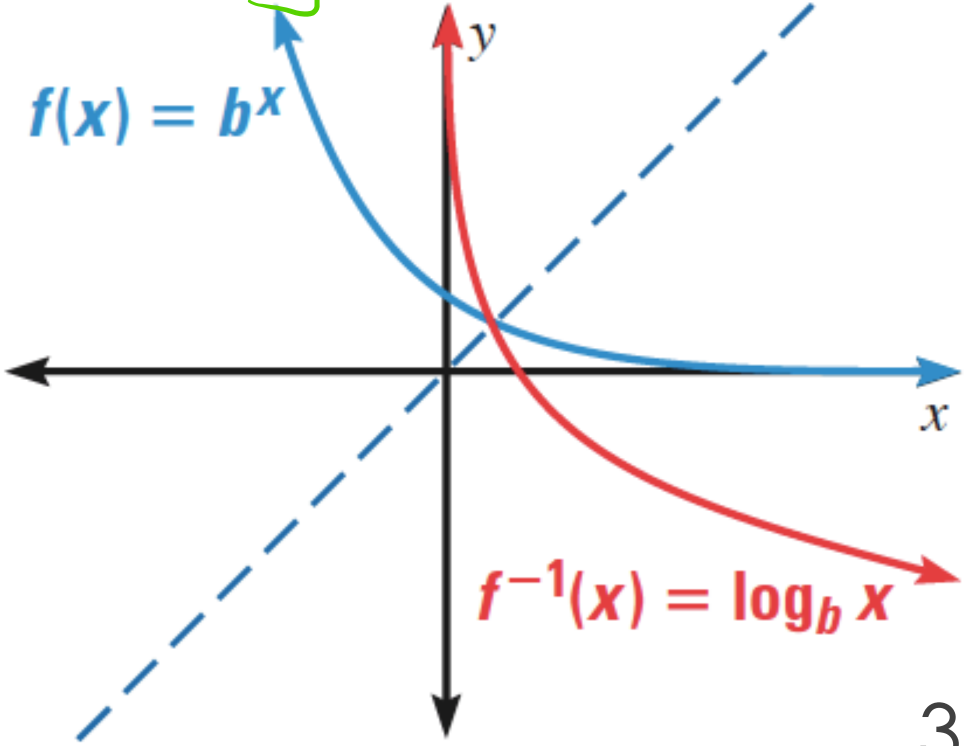
Graphs of f and f^{-1} for $b > 1$

growth functions



Graphs of f and f^{-1} for $0 < b < 1$

decay function



Basic logarithmic function

$$f(x) = log_b x$$

- It has a vertical **asymptote** because zero

exponential functions had horizontal asymptote → log function will have a vertical asymptote.

Dom and range are switched.

Dom: $]h, +\infty[$ or $]-\infty, h[$

Ran : $\mathbb{R}$ (all real numbers)

Effect of "b"

Exponentials
(base)

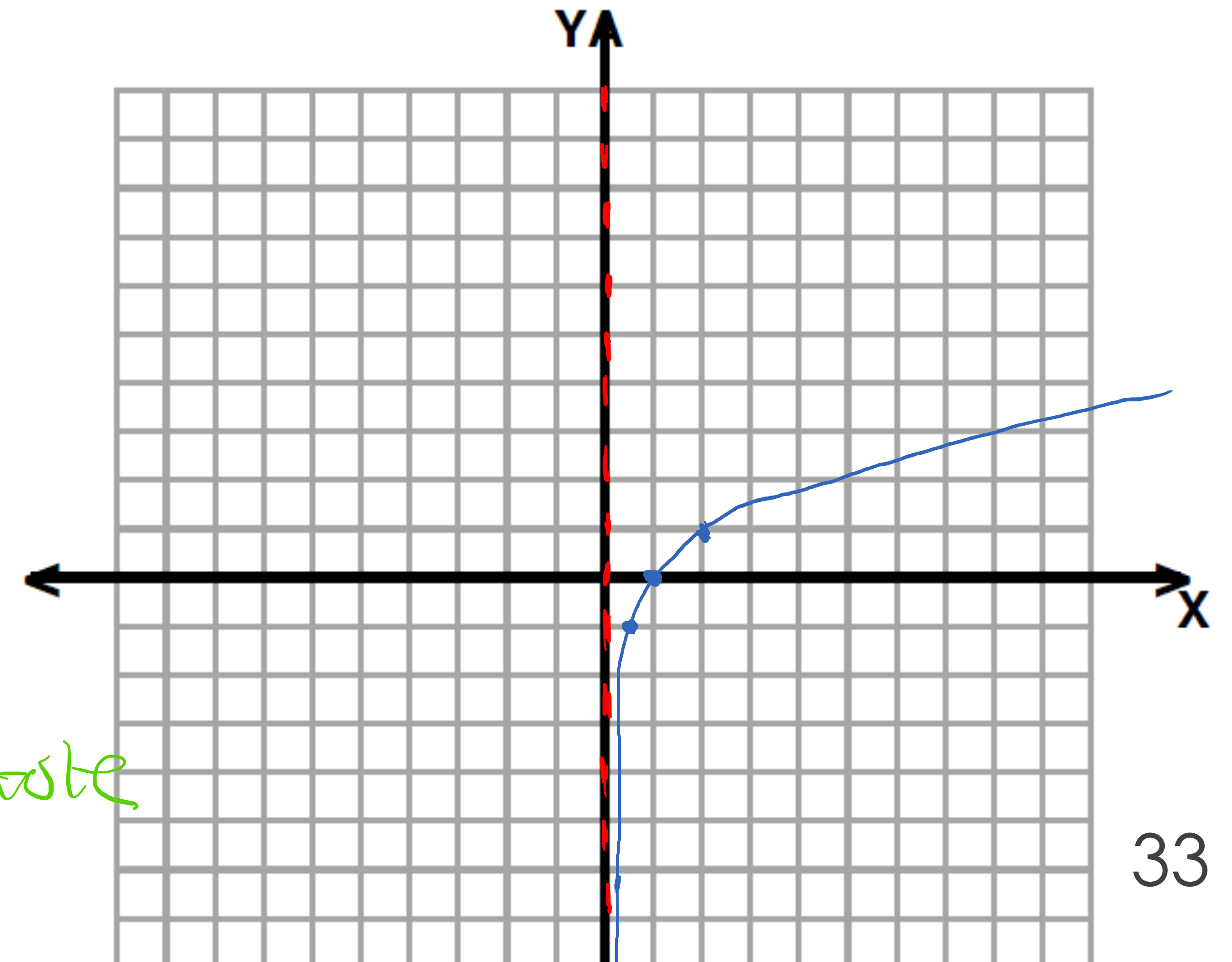
x	-1	0	1
y	1/b	1	b

Graph $f(x) = log_2 x$

	x	$f(x) = \cancel{2^x}$ $\log_2 x$
1/b	1/2	−1
1	1	0
b	2	1

asymptote
$x = 0$

b > 1, the function approaches the asymptote on the bottom.

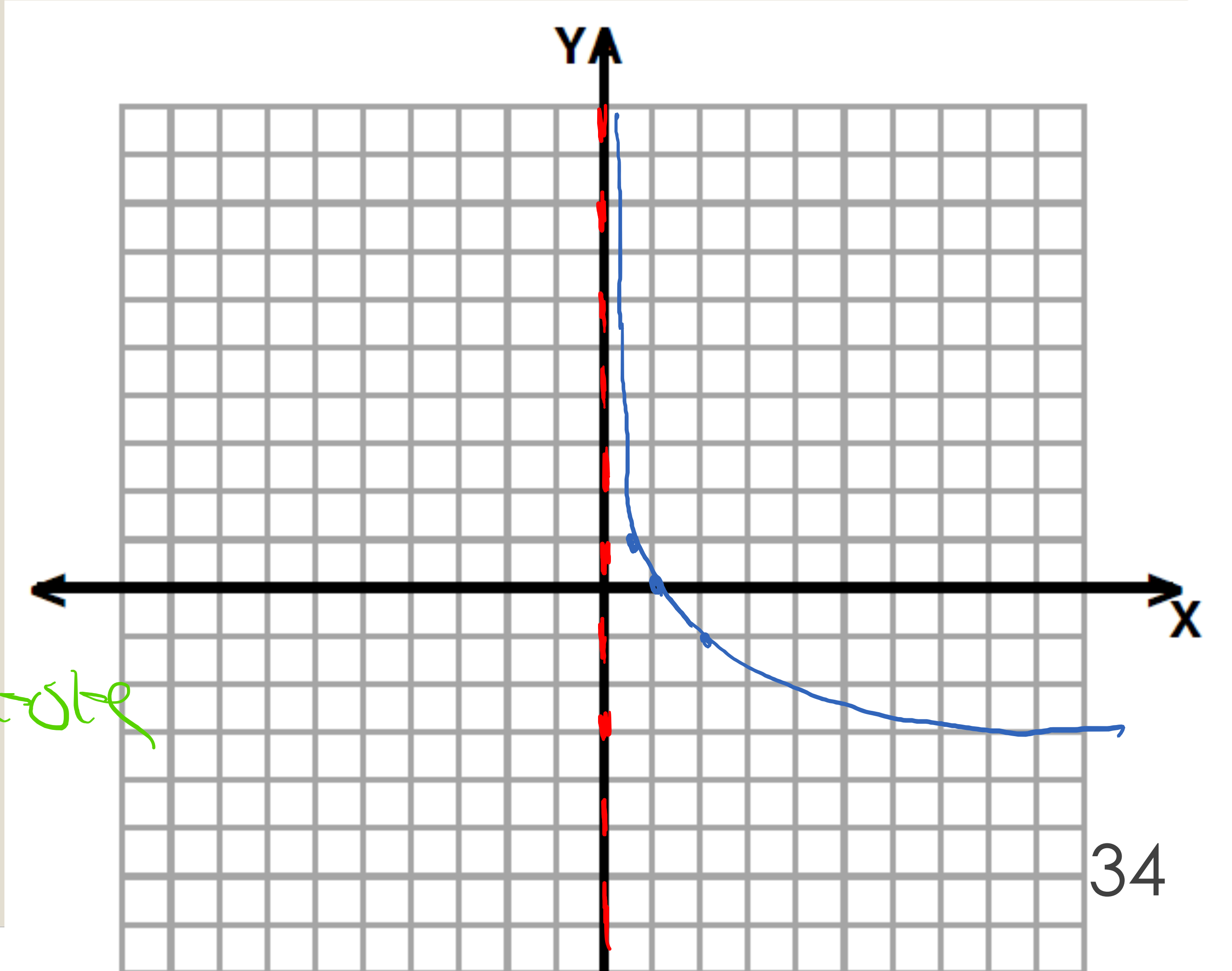


Effect of "b"

Graph $f(x) = log_{\frac{1}{2}}x$

	x	$f(x) = \cancel{2^x}\ \log_{\frac{1}{2}} x$
b	1/2	1
1	1	0
$\frac{1}{b}$	2	-1

$0 < b < 1$ the function approaches the asymptote at the top.



Steps for graphing $f(x) = alog_b(x - h) + k$

1) Create a table of values for the basic functions, using -1,0, and 1 for x.

x	1/b	1	b
y	-1	0	1

2) Apply the transformations using a, h, and k to get the actual points and create a new table of values.

x→ x+h

y→ ay+k

asymptote $x = h$

Parameters "h" and "k"

$\log_{1/3}(x-1)$ h

k

$$y = \log_{1/3}(x) - 1$$

base function $y = \log_{\frac{1}{3}} x$

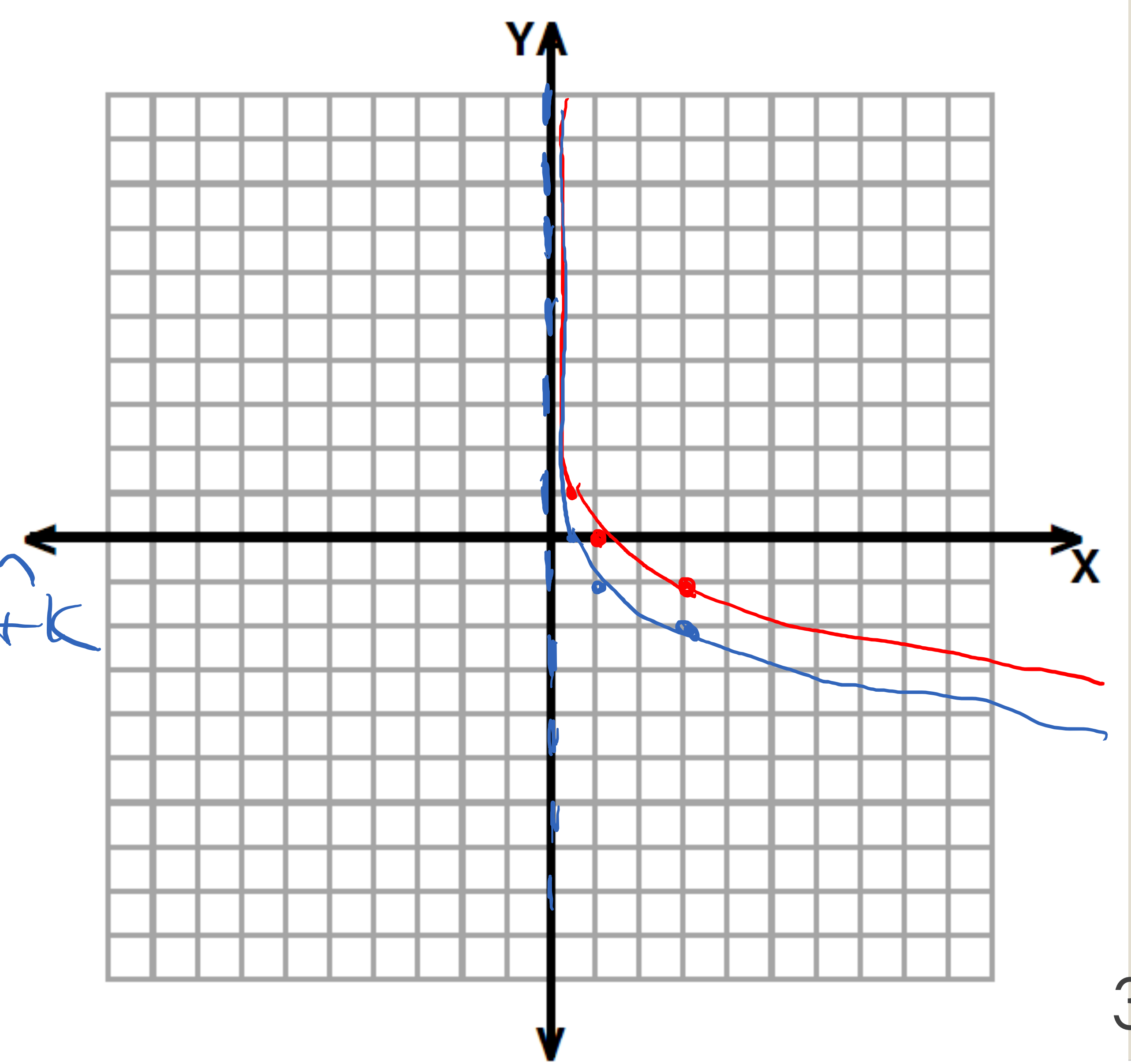
x	3	1	$\frac{1}{3}$
y	-1	0	1

transformation $x \to x+h$

$y \to ay+k$

x	3	1	$\frac{1}{3}$
y	-2	-1	0

asymptote $x = 0$



Parameters "h" and "k

$$y = \log_5(x+2)$$

base function $y = \log_5 x$

x	$\frac{1}{5}$	1	5
y	-1	0	1

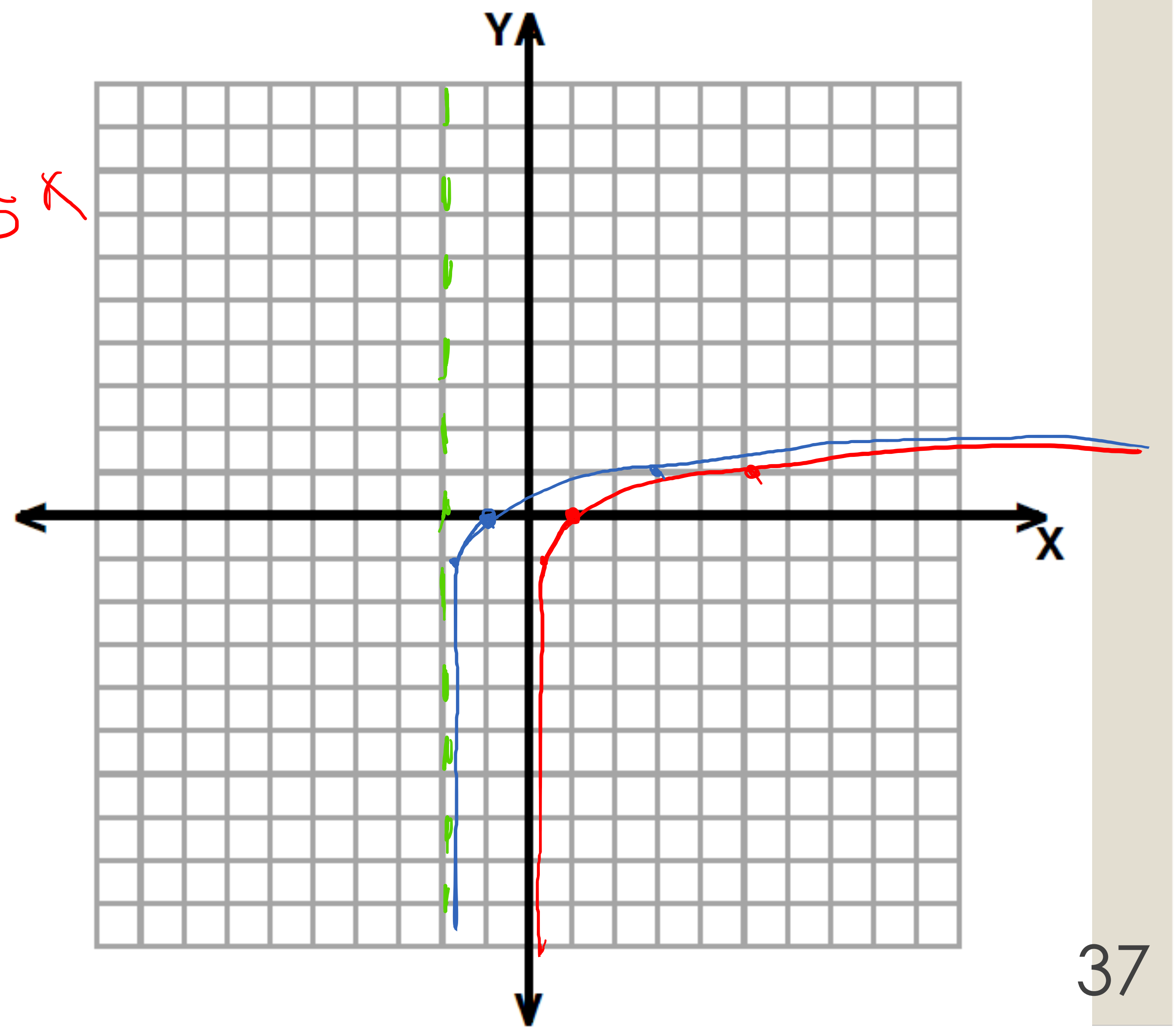
$h = -2$

x	-1.8	-1	3
y	-1	0	1

asymptote

$x = h$

$x = -2$



8.5 – PROPERTIES OF LOGARITHMS

Properties of logarithms

PROPERTIES OF LOGARITHMS

Let b, u, and v be positive numbers such that $b \neq 1$.

PRODUCT PROPERTY $\log_b \boxed{uv} = \log_b u + \log_b v$

ex: $\log_2 (2x)$
$= \log_2 2 + \log_2 x$

QUOTIENT PROPERTY $\log_b \frac{u}{v} = \log_b u - \log_b v$

ex: $\log_2 \frac{5}{3}$
$\log_2 5 - \log_2 3$

POWER PROPERTY $\log_b u^n = n \log_b u$

ex: $\log_3 7^4 = 4 \cdot \log_3 7$

Use $\log_5 3 \approx 0.683$ and $\log_5 7 \approx 1.209$ to approximate the following.

a. $\log_5 \frac{3}{7}$

$= \log_5 3 - \log_5 7$

$= 0.683 - 1.209$

$= -0.527$

b. $\log_5 21$

$= \log_5 (3 \times 7)$

$= \log_5 3 + \log_5 7$

$= 0.683 + 1.209$

$= 1.892$

c. $\log_5 49$

$= \log_5 7^{2}$

$= 2 \cdot \log_5 7$

$= 2 \cdot 1.209$

$= 2.418$

Expanding logs

you know you are "done" when everything inside the log is either a variable w/ no exponents or a prime number.

Expand $\log_2 \frac{7x^3}{y}$. Assume x and y are positive.

$$\log_2 \frac{7x^3}{y} = \underbrace{\log_2(7x^3) - \log_2 y}_{\text{quotient}} = \underbrace{\log_2 7 + \log_2 x^3}_{\text{product}} - \log_2 y$$

$$= \log_2 7 + 3\log_2 x - \log_2 y$$

$\ln 3xy^3$

$\ln 3 + \ln x + \ln y^3$ product

$\ln 3 + \ln x + 3\ln y$

$\log_8 64x^2$

$\log_8 64 + \log_8 x^2$

$\log_8 8^2 + 2\log_8 x$

$2\log_8 8 + 2\log_8 x$

$2\log_8 2^3 + 2\log_8 x$

$6\log_8 2 + 2\log_8 x$

Condensing logs

Condense $\log 6 + 2\log 2 - \log 3$.

$= \log 6 + \log 2^2 - \log 3$

$= \log(6 \cdot 2^2) - \log 3$

$= \log\left(\frac{6 \cdot 2^2}{3}\right) = \log 8$

$7\log_4 2 + 5\log_4 x + 3\log_4 y$

$\log_4 2^7 + \log_4 x^5 + \log_4 y^3$

$\log_4(2^7 \cdot x^5 \cdot y^3)$

$\log_4(2^7 x^5 y^3)$

$3(\ln 3 - \ln x) + (\ln x - \ln 9)$

$3\ln 3 - 3\ln x + \ln x - \ln 9$

$\underbrace{\ln 3^3 - \ln x^3} + \underbrace{\ln x - \ln 9}$

$\ln\left(\frac{3^3}{x^3}\right) + \ln\left(\frac{x}{9}\right)$

$\ln\left(\frac{3^3}{x^3} \cdot \frac{x}{9}\right) = \ln\left(\frac{3}{x^2}\right)$

Change of base

CHANGE-OF-BASE FORMULA

Let u, b, and c be positive numbers with $b \neq 1$ and $c \neq 1$. Then:

$$\log_c u = \frac{\log_b u}{\log_b c}$$

b

$$\log_2 3 = \frac{\log 3}{\log 2}$$

In particular, $\log_c u = \dfrac{\log u}{\log c}$ and $\log_c u = \dfrac{\ln u}{\ln c}$.

This property is useful to plug in logarithms in your calculator, which only has the natural and common logarithms.

Change of base proof (not on test)

$\log_a x = y$ logarithmic form

$a^y = x$ exponential form

$\log_b a^{y} = \log_b x$

$\frac{y \cancel{\log_b a}}{\cancel{\log_b a}} = \frac{\log_b x}{\log_b a}$

$y = \frac{\log_b x}{\log_b a}$

$\log_a x = \frac{\log_b x}{\log_b a}$

Using the change of base formula

Evaluate the expression $\log_3 7$ using common and natural logarithms.

$\log_3 7 = \dfrac{\log 7}{\log 3} = 1.77$

$\log_3 7 = \dfrac{\ln 7}{\ln 3} = 1.77$

$\log_7 12$

$\log_7 12 = \dfrac{\log 12}{\log 7} = 1.277$

$\dfrac{\ln 12}{\ln 7} = 1.277$

8.6 – SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

Types of equations

Exponential	Logarithmic
(1) →Same base, or can get to the same base.	→ Both sides have log with same base. (2)
(3) →Different base. Use logarithms	→ Only one side has a log. Use exponents. (4)

Equal powers property

$2^x = 2^5$

$x = 5$

For $b > 0$ and $b \neq 1$,
if $b^x = b^y$, then $x = y$

Ex: If $3^x = 3^5$ then $x = 5$

Exponential equations: Same base

Examples: same base

turn both sides into the same base

$\cancel{3}^{2x+4} = \cancel{3}^{10}$	$4^{3x-6} = 2^{18}$
$2x+4 = 10$ $-4 \quad -4$ $\frac{2x}{2} = \frac{6}{2}$ $x = 3$	$(2^2)^{3x-6} = 2^{18}$ $\cancel{2}^{2(3x-6)} = \cancel{2}^{18}$ $6x-12 = 18$ $+12 \quad +12$ $\frac{6x}{6} = \frac{30}{6}$ $x = 5$

Recap: Solving exponential equations

1. Rewrite both sides of the equation with a common base. (if needed)
2. Set the exponents on each side equal to each other.
3. Solve the equation.
4. Check your answer.

Equal logarithms property

For positive numbers b, x and y where $b \neq 1$,

$\log_b x = \log_b y$ if and only if $x = y$

Ex: $\log_3 x = \log_3 5$ if and only if $x = 5$

Log equations: same base

Quotient property

$\log_b x - \log_b y$

$= \log_b \frac{x}{y}$

Examples: log of same base on both sides

$\cancel{\log_3}(2x+4) = \cancel{\log_3}(3x+3)$	$\log_5(3x-4) - \log_5(x) = \log_5(2)$
$2x+4 = 3x+3$ $-2x \quad -2x$ $4 = x+3$ $-3 \quad -3$ $1 = x$	$\cancel{\log_5}\left(\frac{3x-4}{x}\right) = \cancel{\log_5} 2$ $x \cdot \frac{3x-4}{x} = 2 \cdot x$ $3x-4 = 2x$ $-3x \quad -3x$ $-4 = -x$ $4 = x$

Recap: Solving Logarithmic equations

1. Rewrite both sides of the equation as one log with common base. (if needed)
2. Set the insides of the log on each side equal to each other.
3. Solve the equation.
4. Check your answer - remove extraneous solutions.

$\cancel{\log_b} \cancel{b}^x = x$

Exponential equations: different base

Examples:

$2^x = 20$	$5^{2x-3} = 16$
$\cancel{\log_2} \cancel{2}^x = \log_2 20$ $x = \log_2 20$ $x = \frac{\log 20}{\log 2}$ $x = 4.32$	$\cancel{\log_5} \cancel{5}^{2x-3} = \log_5 16$ $2x - 3 = \log_5 16$ $+3 \qquad +3$ $2x = \log_5 16 + 3$ $x = \frac{\log_5 16 + 3}{2}$ $x = \frac{\left(\frac{\log 16}{\log 5} + 3\right)}{2}$ $x = 2.36$

Recap: Solving exponential equations with different bases.

1. Take the ~~common~~ log of both sides.
2. Use the power rule to bring the exponent down.
3. Solve the equation.
4. Check your answer.

$\cancel{b}^{\cancel{\log_b} x} = x$

Log equations: log on one side only.

Examples:

$\log_2(4x+3) = 3$	$2\log_3(2x+1) = 4$
$\cancel{2}^{\cancel{\log_2}(4x+3)} = 2^3$ $4x+3 = 8$ $-3 \quad -3$ $\frac{4x}{4} = \frac{5}{4}$ $x = \frac{5}{4}$	$\log_3(2x+1)^2 = 4$ $\cancel{3}^{\log_3(2x+1)^2} = 3^4$ $(2x+1)^2 = 81$ $4x^2+4x+1 = 81$ $4x^2+4x-80 = 0$ $x^2+x-20 = 0$ $(x+5)(x-4) = 0 \quad x=-5 \quad x=4$

Extraneous Solutions

$2\log(2x+1)=4$ cannot be negative.

$x=-5$ $\quad$ $x=4$

$2(-5)+1$ $\quad$ $2(4)+1$

$-10+1$ $\quad$ 9

-9 $\quad$ solution

extraneous

solution $x=4$

Recap: Solving Logarithmic equations

1. Exponentiate both sides.
2. Solve the equation.
3. Check your answer - remove extraneous solutions.